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THRESHOLD PHENOMENA FOR A REACTION-DIFFUSION SYSTEM

David Terman

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

We consider the pure initial value problem for the system of equations

$$v_t = v_{xx} + f(v) - w$$

$$w_t = \varepsilon(v - \gamma w), \quad \varepsilon, \gamma > 0 ,$$

the initial data being $(v(x,0), w(x,0)) = (\varphi(x), 0)$. Here $f(v) = -v + H(v - a)$ where H is the Heaviside step function and $a \in (0, \frac{1}{2})$. This system is of the FitzHugh-Nagumo type, and has several applications including nerve conduction and distributed chemical/biochemical systems. It is demonstrated that this system exhibits a threshold phenomenon. This is done by considering the curve $s(t)$ defined by $s(t) = \sup\{x : v(x,t) = a\}$. The initial datum, $\varphi(x)$, is said to be superthreshold if $\lim_{t \to \infty} s(t) = \infty$. It is proven that the initial datum is superthreshold if $\varphi(x) > a$ on a sufficiently long interval, $\varphi(x)$ is sufficiently smooth, and $\varphi(x)$ decays sufficiently fast to zero as $|x| \to \infty$.

AMS (MOS) Subject Classification: 35K65

Key Words: Superthreshold, Comparison functions

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SIGNIFICANCE AND EXPLANATION

In this paper we consider a system of differential equations which have several applications including nerve conduction and distributed chemical/biochemical systems. Our primary interest is the threshold properties of these equations. This corresponds, for example, to the biological fact that a minimum stimulus is needed to trigger a nerve impulse. One expects that if the initial stimulus is greater than some threshold amount then a signal will be transmitted down the axon. In this case we say that the initial stimulus is superthreshold. We demonstrate that the equations under study do indeed exhibit threshold properties, and we find sufficient conditions for the initial data to be superthreshold.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THRESHOLD PHENOMENA FOR A REACTION-DIFFUSION SYSTEM

David Terman

Section 1. Introduction.

In this paper we consider the pure initial value problem for the system of equations:

$$v_t = v_{xx} + f(v) - w \tag{1.1}$$
$$w_t = \varepsilon(v - \gamma w), \qquad \varepsilon, \gamma > 0 ,$$

the initial data being $(v(x,0),w(x,0)) = (\varphi(x),0)$. We assume that $f(v) = -v + H(v - a)$ where H is the Heaviside step function, and $a \in (0,\frac{1}{2})$. These equations arise as a model for the conduction of electrical impulses in a nerve axon. The most famous such model is due to Hodgkin and Huxley [13], however a mathematical analysis of their model has proven very difficult. The complexity of the Hodgkin and Huxley model led FitzHugh [8] and Nagumo, Arimoto and Yoshizawa [15] to introduce the simpler system (1.1) with $f(v)$ replaced by $f_1(v) = v(1 - v)(v - a)$. The model we consider was introduced as a further simplification by McKean [14]. Surveys of the physiological background of these equations may be found in Cohen [5], Hastings [11], and Rinzel [18].

Here $v = v(x,t)$ represents the electrical potential along the axon as a function of time t and position x, while $w = w(x,t)$ represents a recovery variable needed in order that system (1.1) exhibit pulse shaped solutions. Thinking of $\varphi(x)$ as the initial stimulus, one expects $v(x,t)$ to behave in a manner qualitatively similar to what is observed in the laboratory. For example, electrical impulses in the nerve axon appear to move with constant shape and velocity. Mathematically this corresponds to solutions of the form $(v(x,t),w(x,t)) = (v_c(z),w_c(z))$, $z = x + ct$. Such a solution is often called a traveling wave

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solution. The existence of traveling wave solutions for the FitzHugh-Nagumo system, with $f_1(v)$, was given by Carpenter [3], Conley [4], and Hastings [12] if the parameter ε is sufficiently small. Rinzel and Keller [19] considered the McKean model with $\gamma = 0$. They obtained all of the traveling wave solutions together with their speeds of propagation. Similar results for the McKean model with $\gamma > 0$ have been obtained by Rinzel and Terman [20].

Our primary interest is to study the threshold properties of these equations. That is, if the initial datum, $\varphi(x)$, is sufficiently small, then the solution to (1.1) will decay exponentially fast to zero as $t \to \infty$. This corresponds to the biological fact that a minimum stimulus is needed to trigger a nerve impulse. In this case we say that $\varphi(x)$ is subthreshold. One expects, however, that if the initial stimulus is sufficiently large, or superthreshold, then a signal will be transmitted down the axon. We show this to be the case if the parameter ε is chosen sufficiently small.

Throughout this paper we assume that the initial datum $\varphi(x)$ satisfies

(a) $\varphi(x) \in C^1(\mathbb{R})$

(b) $\varphi(x) \in [0,1]$

(c) $\varphi(x) = \varphi(-x)$ in $\mathbb{R}$

(1.2) (d) There exists a unique constant $x_0 > 0$ such that $\varphi(x_0) = a$

(e) $\varphi(x) > a$ for $|x| < x_0$

(f) $|\varphi(x)| < ae^{\frac{\sqrt{2}}{2}(x_0 - x)}$ for $|x| > x_0$

(g) φ'' is a bounded continuous function except possibly when $|x| = x_0$.

This last condition is needed in order to obtain sufficient a priori bounds on the derivatives of the solution of system (1.1).

Note that in some sense x_0 determines the size of the initial datum. Therefore, we expect a signal to propagate if x_0 is sufficiently large. In order to be more precise we consider the curve $s(t)$ given by $s(t) = \sup\{x : v(x,t) = a\}$.

We define $\varphi(x)$ to be <u>superthreshold</u> if $s(t)$ is defined for all $t \in R^+$ and $\lim_{t\to\infty} s(t) = \infty$.

Note that because $f(v)$ is discontinuous we cannot expect the solution (v,w) to be very smooth. By a classical solution of system (1.1) we mean the following.

<u>Definition</u>: Let $S_T = R \times (0,T)$ and $G_T = \{(x,t) \in S_T: v(x,t) \neq a\}$. Then $(v(x,t),w(x,t))$ is said to be a classical solution of the Cauchy problem (1.1) in S_T if

(a) (v,w) along with (v_x,w_x) are bounded continuous functions in S_T,

(b) in G_T, v_{xx}, v_t and w_t are continuous functions which satisfy the system of equations

$$v_t = v_{xx} + f(v) - w$$

$$w_t = \varepsilon(v - \gamma w) \ ,$$

(c) $\lim_{t\downarrow 0} v(x,t) = \varphi(x)$ and $\lim_{t\downarrow 0} w(x,t) = 0$ for each $x \in R$.

Throughout this paper we assume that there exists a unique classical solution of the Cauchy problem (1.1) in $R \times R^+$. In particular, we assume that there exist constants V and W such that $|v(x,t)| < V$ and $|w(x,t)| < W$ in $R \times R^+$. Using the method of invariant rectangles (see Weinberger [25]) Rauch and Smoller [17] considered system (1.1) with $f_1(v)$ and showed that such bounds exist if

$$\lim_{|v|\to\infty} \left|\frac{f_1(v)}{v}\right| > \frac{1}{\gamma} \ . \tag{1.3}$$

A similar argument shows that such bounds exist for system (1.1) with $f(v)$ if $f(v)$ satisfies (1.3). The results of this paper are still valid if we assume that $f(v) = -v + H(v - a)$ for $|v| \leq 1$, $f(v)$ is a smooth function with $f'(v) < 0$ for $|v| > 1$, and $\lim_{|v|\to\infty} \left|\frac{f(v)}{v}\right| > \frac{1}{\gamma}$. In [23] it is shown that a classical solution of the Cauchy problem (1.1) does exist in S_T for some $T > 0$.

It will also be necessary to assume that the curve $s(t)$ does not behave too wildly. These assumptions are described in the second paragraph of Section 3.

With these assumptions we prove the following result:

Theorem 1.1: Choose $a \in (0,\frac{1}{2})$ and $\gamma > 0$. Then there exist positive constants δ and θ such that if $\epsilon \in (0,\delta)$ and $\varphi(x)$ satisfies (1.2) with $x_0 > \theta$, then $\varphi(x)$ is superthreshold.

In this paper we only find sufficient conditions for the initial data to be superthreshold. Subthreshold results for the FitzHugh-Nagumo model were given by Rauch and Smoller [17]. Using a linearization technique they showed that the rest state $(v,w) = (0,0)$ is stable in an appropriate Hilbert space. For some values of the parameters ϵ and γ they were also able to prove the stability of the rest state by constructing a Lyapunov function. Further subthreshold results were given by Schonbek [21].

The primary techniques used throughout this paper are comparison theorems for parabolic partial differential equations. These results are discussed in Section 2.

In Section 2 we also state some results about solutions of the scalar equation

$$(1.4) \qquad \begin{aligned} v_t &= v_{xx} + f(v) \\ v(x,0) &= \varphi(x) . \end{aligned}$$

These results are proven in [22], and will play a very important role in the proof of Theorem 1.1. Equations similar to (1.4) have many applications and have been studied by a number of people (see [2] for references). R. A. Fisher [7] introduced equation (1.4) with $f(v) = v(1 - v)$ in connection with certain problems in population genetics. Other applications occur in theories of combustion and active transmission lines. Threshold results for this scalar equation were given by Aronson and Weinberger [2] who made the following assumptions on $f(v)$:

$$f \in C^1[0,1],\ f(0) = f(1) = 0,\ f'(0) < 0 ,$$

$$(1.5) \qquad f(v) < 0 \text{ in } (0,a),\ f(v) > 0 \text{ in } (a,1) \text{ for some } a \in (0,1), \text{ and}$$

$$\int_0^1 f(v)dv > 0 .$$

In their paper, Aronson and Weinberger referred to this as the "heterozygote inferior" case in connection with the Fisher model for population genetics. They showed that if the initial datum, $\varphi(x)$, is sufficiently small, then $\lim_{t\to\infty} v(x,t) = 0$, while if $\varphi(x)$ is sufficiently large on a sufficiently large interval, then $\lim_{t\to\infty} v(x,t) = 1$ for $x \in R$. Here we briefly discuss the proof of the superthreshold part of their results because it clearly illustrates how comparison functions are constructed and why one expects equation (1.4), or system (1.1), to exhibit a threshold phenomenon.

Aronson and Weinberger construct comparison functions $q_\eta(x)$ which are solutions of the steady state equation

$$q_\eta'' + f(q_\eta) = 0 \; . \tag{1.6}$$

The phase plane configuration of this equation is shown in Figure 1.1.

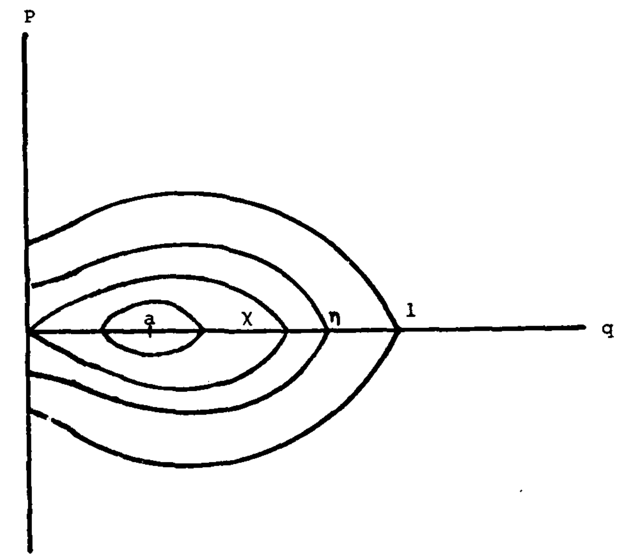

Figure 1.1

Because of the assumptions on f there exists a unique constant $\chi \in (0,1)$ such that $\int_0^\chi f(u)du = 0$. If $\eta \in (\chi,1)$ then the trajectory through $(\eta,0)$ is characterized by $\frac{1}{2}p^2 + F(q) = F(\eta)$. It is easy to show that this trajectory intersects both the positive and negative p axis. Thus, given $\eta \in (\chi,1)$, there exists a $b_\eta \in R^+$ and a function $q_\eta(x)$ such that $q_\eta(x)$ is a solution of the steady state equation (1.6) which satisfies

$$0 = q_\eta(\pm b_\eta) < q_\eta(x) = q_\eta(0) \quad \text{in} \quad (-b_\eta, b_\eta) \ .$$

In fact, one can show that $b_\eta = 2\int_0^\eta \{2F(\eta) - 2f(y)\}^{-1/2} dy$ where $F(y) = \int_0^y f(u)du$.

Aronson and Weinberger then prove the following result.

<u>Theorem</u>: Let $v(x,t) \in [0,1]$ be a solution of (1.4) in $\mathbb{R} \times \mathbb{R}^+$ where $f(v)$ satisfies (1.5). If $\varphi(x) \geq q_\eta(\chi - x_0)$ in $(x_0 - b_\eta, x_0 + b_\eta)$ for some $\eta \in (\chi,1)$ and for some $x_0 \in \mathbb{R}$, then

$$\lim_{t \to +\infty} u(x,t) = 1 \ .$$

In this construction we see that a sufficient condition for the initial datum to be superthreshold is that it lie above one of the comparison functions $q_\eta(x)$, $\eta \in (\chi,1)$. We expect equation (1.4) to exhibit a threshold phenomenon because comparison functions of arbitrarily small length or height cannot be constructed.

Further results for the scalar equation (1.4) were given by Fife and McLeod [6]. Assuming that $f(v)$ satisfies (1.5) with $f'(1) < 0$, they showed that if $\varphi(x)$ is superthreshold then the solution will converge uniformly to a traveling wave solution.

The proof of Theorem 1.1 is given in Section 3.

Section 2. Preliminary Results

The principal tools used in the proof of Theorem (1.1) are presented in this section. We begin by stating, without proof, some standard comparison theorems for linear parabolic equations. Proofs of the comparison theorems may be found in Protter and Weinberger [16]. We then prove a comparison theorem for solutions of a nonlinear system of equations. Finally, we discuss those results which are needed about the scalar equation:

$$v_t = v_{xx} + f(v) \quad (2.1)$$

Throughout this paper we will constantly be comparing solutions of the system (1.1) to solutions of the scalar equation (2.1).

Comparison Theorems

For $T > 0$ let $\alpha_1(t)$ and $\alpha_2(t)$ be continuous functions on $[0,T]$. We assume that $\alpha_1(t)$ is either finite or identically $-\infty$ and $\alpha_2(t)$ is either finite or identically $+\infty$ on $[0,T]$. Let

$$D = \{(x,t): \alpha_1(t) < x < \alpha_2(t),\ 0 < t < T\}$$

and let L be the linear operator defined by

$$Lu = u_t - u_{xx} + u .$$

Assume that in D, $u(x,t)$ is a bounded continuous function with u_x, u_{xx}, and u_t continuous.

Theorem 2.1: Assume that $u(x,t)$ satisfies the inequalities

(2.2) (a) $Lu > 0$ in D,

(b) $u(x,0) \geq 0$ in $[\alpha_1(0), \alpha_2(0)]$.

If $\alpha_1(t)$ or $\alpha_2(t)$ are finite assume that

(c) $u(\alpha_1(t),t) \geq 0$ in $[0,T]$,

(d) $u(\alpha_2(t),t) \geq 0$ in $[0,T]$.

Then $u(x,t) \geq 0$ in D. If $u(x,0) > 0$ for some $x \in (\alpha_1(0), \alpha_2(0))$, then $u(x,t) > 0$ in D.

Theorem 2.2: Assume that $\alpha_k(t)$ is finite for either $k = 1$ or 2. Furthermore, assume that $\alpha_k(t) \in C^1([0,T])$, and $u(x,t)$ satisfies the inequalities (2.2a-d) with $u(x,0) > 0$ for some $x \in (\alpha_1(0),\alpha_2(0))$. If $u(\alpha_k(t),t) = 0$ for some $t \in (0,T]$, then $(-1)^k u_x(\alpha_k(t),t) < 0$.

Theorem 2.3: Assume that $u(x,t)$ satisfies the inequalities (2.2a,b) with $u(x,0) > 0$ on $[\alpha_1(0),\alpha_2(0)]$. If, for $k = 1$ or 2, $\alpha_k(t)$ is finite, assume that $\alpha_k(t) \in C^1((0,T))$ and $(-1)^k u_x(\alpha_k(t),t) > 0$ on $(0,T)$. Then $u(x,t) > 0$ in $\bar{D}$.

The Scalar Equation

The following result is proven in [22, Corollary 6.4].

Theorem 2.4: Choose $a \in (0,\frac{1}{2})$ and $K < \frac{1}{2} - a$. Assume that $d < 1 - K$ and $r_1 > 0$. Furthermore, assume that $\psi(x)$ satisfies (1.2), and $u(x,t)$ is the solution of the equation

$$u_t = u_{xx} + f(u) - K \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+ ,$$
$$u(x,0) = \psi(x) \qquad \text{in} \quad \mathbb{R} .$$

Then there exist constants θ, r, and T, which depend only on a, K, and d such that if $x_0 > \theta$, then $v(x,t) > d$ for $|x| < r_1$, $t > T$. Furthermore, the curve $s(t)$, given by $s(t) = \sup\{x : v(x,t) = a\}$, is a well defined, continuously differentiable function which satisfies

(a) $s'(t)$ is a locally Lipschitz continuous function

(b) $\lim_{t\to\infty} s(t) = \infty$

(c) $s(t) > x_0 - r$ for $t \in \mathbb{R}^+$.

A Comparison Theorem for Systems of Equations

Suppose that for $T > 0$ the functions $v_0(x,t)$ and $v_1(x,t)$ satisfy $v_k(x,t) \in C^2(\mathbb{R}^+ \times (0,T))$, $k = 1,2$, and $v_1(x,t) < v_0(x,t)$ in $\mathbb{R}^+ \times (0,T)$. Furthermore, suppose that $\alpha_0(x)$ and $\alpha_1(x)$ are smooth functions which satisfy $\alpha_0(x) < \alpha_1(x)$

in $\mathbf{R}^+$. Let $w_k(x,t)$, $k = 0,1$, be solutions of the ordinary differential equations:

$$w_{kt} = \varepsilon(v_k - \gamma w_k) \quad \text{in} \quad \mathbf{R}^+ \times (0,T) ,$$
$$w_k(x,0) = \alpha_k(x) \qquad \text{in} \quad \mathbf{R}^+ .$$

Note that $w_0(x,t) < w_1(x,t)$ in $\mathbf{R}^+ \times (0,T)$. Let L_1 be the operator defined by $L_1 u \equiv u_t - u_{xx} - g(u)$ where g is a bounded, smooth function. Finally, let (v,w) be the solution of the system of equations

$$\text{(2.3)} \qquad \begin{cases} L_1 v = -w \\ w_t = \varepsilon(v - \gamma w) \quad \text{in} \quad \mathbf{R}^+ \times (0,T) , \end{cases}$$

with initial and boundary conditions

$$\text{(2.3a)} \qquad (v(x,0),w(x,0)) = (\varphi(x),\psi(x)) \quad \text{in} \quad \mathbf{R}^+$$
$$v(0,t) = h(t) \qquad \text{in} \quad (0,T) .$$

We assume that the functions $\varphi(x)$, $\psi(x)$, and $h(t)$ are all continuously differentiable.

<u>Proposition 2.5</u>: *Assume that* $v_1(x,0) \leq \varphi(x) \leq v_0(x,0)$, $v_1(0,t) \leq h(t) \leq v_0(0,t)$, *and* $\alpha_1(x) \leq \psi(x) \leq \alpha_0(x)$ *in* $\mathbf{R}^+$. *Furthermore, assume that*

$$\text{(2.4)} \qquad \begin{aligned} &L_1 v_0 \geq -w_1 , \\ &L_1 v_1 \leq -w_0 \quad \text{in} \quad \mathbf{R}^+ \times (0,T) . \end{aligned}$$

Then, $v_1(x,t) \leq v(x,t) \leq v_0(x,t)$, *and* $w_1(x,t) \leq w(x,t) \leq w_0(x,t)$ *in* $\mathbf{R}^+ \times (0,T)$.

<u>Proof</u>: Using an iteration scheme, we approximate the solution of (2.3) by a sequence of functions $(v_n(x,t),w_n(x,t))$. We show that for each $n \geq 2$, $v_1(x,t) \leq v_n(x,t) \leq v_0(x,t)$ and $w_1(x,t) \leq w_n(x,t) \leq w_0(x,t)$ in $\mathbf{R}^+ \times (0,T)$. We then show that the sequence of functions (v_n,w_n) converges uniformly on bounded sets to the solution (v,w).

The functions $v_n(x,t)$ and $w_n(x,t)$ are defined as follows. Suppose that for $n \geq 2$, $v_0,v_1,\ldots,v_{n-1}$ and $w_0,w_1,\ldots,w_{n-1}$ have been defined. We then let $v_n(x,t)$ be the solution of the equations:

$$
(2.5) \qquad \begin{cases} L_1 v_n = -w_{n-1} & \text{in } \mathbb{R}^+ \times (0,T) , \\ v_n(x,0) = \varphi(x) & \text{in } \mathbb{R}^+ , \\ v_n(0,t) = h(t) & \text{in } (0,T) . \end{cases}
$$

We then let $w_n(x,t)$ be the solution of the equations:

$$
(2.6) \qquad \begin{aligned} w_{nt} &= \varepsilon(v_n - \gamma w_n) \quad \text{in } \mathbb{R}^+ \times (0,T) , \\ w_n(x,0) &= \psi(x) \quad \text{in } \mathbb{R}^+ . \end{aligned}
$$

We show, using induction, that for $n > 2$,

$$
(2.7a) \qquad (-1)^n v_{n-1}(x,t) < (-1)^n v_n(x,t) < (-1)^n v_{n-2}(x,t)
$$

and,

$$
(2.7b) \qquad (-1)^n w_{n-1}(x,t) < (-1)^n w_n(x,t) < (-1)^n w_{n-2}(x,t) \quad \text{in } \mathbb{R}^+ \times (0,T) .
$$

First, suppose that $n = 2$. We wish to show that $v_1(x,t) < v_2(x,t) < v_0(x,t)$ in $\mathbb{R}^+ \times (0,T)$. This is proven by a comparison argument. Note that, by our assumptions, $v_1(x,0) < v_2(x,0) < v_0(x,0)$ and $v_1(0,t) < v_2(0,t) < v_0(0,t)$. Furthermore, $L_1 v_1 < L_1 v_2 < L_1 v_0$ in $\mathbb{R}^+ \times (0,T)$. This is because $L_1 v_2 = -w_1$, and we are assuming that $-w_1 < L_1 v_0$ and $-w_1 > -w_0 > L_1 v_1$ in $\mathbb{R}^+ \times (0,T)$. From Theorem 2.1 of [2] it follows that $v_1 < v_2 < v_0$ in $\mathbb{R}^+ \times (0,T)$. It immediately follows from (2.6) that $w_1 < w_2 < w_0$ in $\mathbb{R}^+ \times (0,T)$.

Now suppose that, for $n > 2$, (2.7a) and (2.7b) hold. If n is even, we conclude from (2.7b) that

$$
w_{n-1}(x,t) < w_n(x,t) < w_{n-2}(x,t) \quad \text{in } \mathbb{R}^+ \times (0,T) .
$$

From (2.5) it follows that

$$
L_1(v_{n-1}) < L_1(v_{n+1}) < L_1(v_n) \quad \text{in } \mathbb{R}^+ \times (0,T).
$$

Since $v_{n-1}(x,0) = v_{n+1}(x,0) = v_n(x,0) = \varphi(x)$ and $v_{n-1}(0,t) = v_{n+1}(0,t) = v_n(0,t) = h(t)$ it follows from Theorem 2.1 of [2] that

$$
(2.8a) \qquad v_{n-1}(x,t) < v_{n+1}(x,t)) < v_n(x,t) \quad \text{in } \mathbb{R}^+ \times (0,T) .
$$

If n is odd, then from (2.7b) and (2.5) it follows that

$$
L_1(v_n) < L_1(v_{n+1}) < L_1(v_{n-1}) \quad \text{in } \mathbb{R}^+ \times (0,T).
$$

Since $v_n(x,0) = v_{n+1}(x,0) = v_{n-1}(x,0) = \varphi(x)$ and $v_n(0,t) = v_{n+1}(0,t) = v_{n-1}(0,t) = h(t)$ we conclude that

(2.8b) $$v_n(x,t) \le v_{n+1}(x,t) \le v_{n-1}(x,t) \quad \text{in} \quad \mathbb{R}^+ \times (0,T) .$$

Combining (2.8a) and (2.8b) it follows that

$$(-1)^{n+1}v_n(x,t) \le (-1)^{n+1}v_{n+1}(x,t) \le (-1)^{n+1}v_{n-1}(x,t) \quad \text{in} \quad \mathbb{R}^+ \times (0,T)$$

and using (2.6),

$$(-1)^{n+1}w_n(x,t) \le (-1)^{n+1}w_{n+1}(x,t) \le (-1)^{n+1}w_{n-1}(x,t) \quad \text{in} \quad \mathbb{R}^+ \times (0,T) .$$

This completes the induction argument that (2.7) holds.

We have now shown that

$$v_1 \le v_3 \le \cdots \le v_{2n+1} \le \cdots \le v_{2n} \le \cdots \le v_2 \le v_0$$

and

$$w_1 \le w_3 \le \cdots \le w_{2n+1} \le \cdots \le w_{2n} \le \cdots \le w_2 \le w_0$$

in $\mathbb{R}^+ \times (0,T)$. Hence, there exist pairs of functions $(\bar{v},\bar{w})$ and $(\hat{v},\hat{w})$ such that (v_{2n},w_{2n}) converges to $(\bar{v},\bar{w})$ and (v_{2n+1},w_{2n+1}) converges to $(\hat{v},\hat{w})$ uniformly on bounded sets of $\mathbb{R}^+ \times (0,T)$ as $n \to \infty$. Clearly, $v_1 \le \bar{v} \le v_0$, $v_1 \le \hat{v} \le v_0$, $w_1 \le \bar{w} \le w_0$ and $w_1 \le \hat{w} \le w_0$ in $\mathbb{R}^+ \times (0,T)$. To complete the proof of the Proposition we show that $(v,w) \equiv (\bar{v},\bar{w}) \equiv (\hat{v},\hat{w})$ in $\mathbb{R}^+ \times (0,T)$.

Let $G(x,\xi,t,\tau)$ be the Green's function for the domain $\mathbb{R}^+ \times \mathbb{R}^+$. That is,

$G(x,\xi,t,\tau) = K(x-\xi,t-\tau) - K(x+\xi,t-\tau)$ where $K(x,t) = \frac{1}{2\pi^{1/2}t^{1/2}} e^{-\frac{x^2}{4t}}$. Then, for $n \ge 1$,

$$v_{2n}(x,t) = \int_0^\infty G(x,\xi,t,\tau)\varphi(\xi)d\xi + \int_0^t h(\tau)G_\xi(x,0,t,\tau)d\tau$$

(2.9) $$+ \int_0^t \int_0^\infty G(x,\xi,t,\tau)[g(v_{2n}(\xi,\tau)) - w_{2n-1}(\xi,\tau)]d\xi d\tau ,$$

$$w_{2n}(x,t) = e^{-\varepsilon\gamma t}\psi(x) + \varepsilon e^{-\varepsilon\gamma t}\int_0^t e^{-\varepsilon\gamma\eta}v_{2n}(x,\eta)d\eta .$$

Passing to the limit in (2.9) it follows that

$$\bar{v}(x,t) = \int_0^\infty G(x,\xi,t,\tau)\varphi(\xi)d\xi + \int_0^t h(\tau)G_\xi(x,0,t,\tau)d\tau$$

$$+ \int_0^t \int_0^\infty G(x,\xi,t,\tau)[g(v(\xi,\bar{\tau})) - \hat{w}(\xi,\tau)]d\xi d\tau ,$$

$$\bar{w}(x,t) = e^{-\varepsilon\gamma t}\psi(x) + \varepsilon e^{-\varepsilon\gamma t}\int_0^t e^{\varepsilon\gamma\eta}\bar{v}(x,\eta)d\eta .$$

Hence $(\bar{v},\bar{w})$, $(\bar{v}_x,\bar{w}_x)$, $(\bar{v}_{xx},\bar{w}_{xx})$, and $(\bar{v}_t,\bar{w}_t)$ are all continuous functions and $(\bar{v},\bar{w})$ solves the system of equations:

$$\text{(2.10a)} \qquad \begin{cases} L_1(\bar{v}) = -\hat{w} \\ \bar{w}_t = \varepsilon(\bar{v} - \gamma\bar{w}) \quad \text{in} \quad R^+ \times (0,T) , \end{cases}$$

$$(\bar{v}(x,0),\bar{w}(x,0)) = (\varphi(x),\psi(x)) \quad \text{in} \quad R^+ ,$$

$$\bar{v}(0,t) = h(t) \qquad \text{in} \quad (0,T) .$$

Similarly, $(\hat{v},\hat{w})$ solves the equations:

$$\text{(2.10b)} \qquad \begin{cases} L_1\hat{v} = -\bar{w} \\ \hat{w}_t = \varepsilon(\hat{v} - \gamma\hat{w}) \quad \text{in} \quad R^+ \times (0,T) , \end{cases}$$

$$(\hat{v}(x,0),\hat{w}(x,0)) = (\varphi(x),\psi(x)) \quad \text{in} \quad R^+ ,$$

$$\hat{v}(0,t) = h(t) \qquad \text{in} \quad (0,T) .$$

Let $z(x,t) = \bar{v}(x,t) - \hat{v}(x,t)$ and $u(x,t) = \bar{w}(x,t) - \hat{w}(x,t)$. 'Subtracting' (2.10b) from (2.10a) we conclude that (z,y) is a solution of the equations:

$$\begin{cases} z_t = z_{xx} + g(\bar{v}) - g(\hat{v}) + y , \\ y_t = \varepsilon(z - \gamma y) \qquad \text{in} \quad R^+ \times (0,T) , \end{cases}$$

$$(z(x,0),y(x,0)) = (0,0) \quad \text{in} \quad R^+ ,$$

$$z(0,t) = 0 \qquad \text{in} \quad (0,T) .$$

From the mean value theorem and our assumptions on g, there exists a bounded function $\beta(x,t)$ such that $g(\bar{v}) - g(\hat{u}) = \beta(x,t)(\bar{v} - \hat{u})$. Hence, (z,y) solves the linear system of equations:

$$
(2.11) \qquad \begin{cases} z_t = z_{xx} + \beta(x,t)z + y \\ y_t = \varepsilon(z - \gamma y) \qquad \text{in } \mathbb{R}^+ \times (0,T) , \end{cases}
$$

$$
(z(x,0),y(x,0)) = (0,0) \quad \text{in } \mathbb{R}^+ ,
$$

$$
z(0,t) = 0 \quad \text{in } (0,T) .
$$

We wish to show that $(z,y) \equiv (0,0)$ in $\mathbb{R}^+ \times (0,T)$. This would imply that $(\bar{v},\bar{w}) \equiv (\hat{v},\hat{w})$.

Note that (z,y) can be written implicitly as a solution of the integral equations:

$$
(2.11a) \qquad z(x,t) = \int_0^t \int_0^\infty G(x,\xi,t,\tau)[\beta(\xi,\tau)z(\xi,\tau) + y(\xi,\tau)]d\xi d\tau ,
$$

$$
(2.11b) \qquad y(x,t) = \varepsilon e^{-\varepsilon\gamma t} \int_0^t e^{\gamma\varepsilon\eta} z(x,\eta)d\eta .
$$

Let

$$
\rho(t) = \sup_{x\in\mathbb{R}^+} \{|z(x,t)| + |y(x,t)|\} ,
$$

and suppose that $|\beta(x,t)| < B$ in $\mathbb{R}^+ \times (0,T)$. Taking $B > 1$ it follows from (2.11a) that

$$
(2.12a) \qquad |z(x,t)| \le B \int_0^t \rho(\tau)d\tau .
$$

From (2.11b) we conclude that

$$
(2.12b) \qquad |y(x,t)| \le \varepsilon \int_0^t \rho(\tau)d\tau .
$$

Adding (2.12a) and (2.12b) we obtain

$$\rho(t) \leq (B + \varepsilon) \int_0^t \rho(\tau)d\tau \qquad \text{in} \quad (0,T) .$$

From Gronwall's inequality we now obtain that $\rho(t) = 0$ in $(0,T)$. Hence $(z,y) \equiv (0,0)$ and $(\bar{v},\bar{w}) \equiv (\hat{v},\hat{w})$ in $\mathbb{R}^+ \times (0,T)$. It now follows from (2.10a) that $(\bar{v},\bar{w})$ is a solution of the system (2.3). However, an argument similar to the one just given shows that the solution to system (2.3) is unique, and, therefore, $(v,w) \equiv (\bar{v},\bar{w})$ in $\mathbb{R}^+ \times (0,T)$.

Section 3. Proof of Theorem 1.1

Throughout this section we assume that the initial datum, $\varphi(x)$, satisfies the assumptions (1.2), and there exists a unique classical solution of the Cauchy problem (1.1) in $\mathbb{R} \times \mathbb{R}^+$. Because of the assumption (1.2), the pure initial value problem (1.1) is equivalent to the initial-boundary value problem:

$$(3.1)\qquad \begin{cases} v_t = v_{xx} + f(v) - w \\ w_t = \varepsilon(v - \gamma w) \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^+ , \end{cases}$$

$$(v(x,0),w(x,0)) = (\varphi(x),0) \quad \text{in } \mathbb{R}^+ ,$$

$$v_x(0,t) = 0 \qquad \text{in } \mathbb{R}^+ .$$

Throughout the rest of this paper we only consider solutions of this quarter plane problem.

Recall that $\varphi(x)$ is said to be superthreshold if $\lim_{t\to\infty} s(t) = \infty$ where $s(t) = \sup\{x: v(x,t) = a\}$. In the introduction we mentioned that it would be necessary to make some assumptions on the function $s(t)$. We now describe these assumptions.

Assumptions in $s(t)$.

If $s(t)$ is continuous in the interval (t_0,t_1), then $\lambda = \lim_{t\uparrow t_1} s(t)$ exists. Furthermore, there exist positive constants M and δ such that for $t_1 - \delta < t < t_1$ either

$$(3.2)\qquad \begin{aligned} &\text{(a)} \quad s(t) < -M(t - t_1) + \lambda , \\ \text{or} \quad &\text{(b)} \quad s(t) > M(t - t_1) + \lambda . \end{aligned}$$

Note that these conditions are satisfied if $s(t)$ does not approach λ tangentially from both directions as $t \uparrow t_1$. In particular, they are satisfied if $s(t)$ does not change directions infinitely often in every neighborhood of $t = t_1$.

The principal tools used in the proof of Theorem 1.1 are the comparison theorems. We construct two, one-parameter families of comparison functions $G_{x_0}(x)$ and $H_{x_0}(x)$, defined for $x_0 > \theta$ and $x > 0$, which serve, respectively, as a lower bound for v and an upper bound for w. The proof of Theorem 1.1 is then split into two parts. We first prove the following.

Theorem 3.1: Fix $a \in (0,\frac{1}{2})$ and $\gamma > 0$. There exist positive constants δ, θ, T_1, T_2, and λ_1 with the following properties. Assume that $\varepsilon \in (0,\delta)$ and $\varphi(x)$ satisfies the assumptions (1.2) with $x_0 > \theta$. Furthermore, assume that

$$\text{(3.3a)}\qquad \begin{aligned} &\text{(a)}\quad \varphi(x) > G_{x_0}(x) && \text{in } [0,x_0)\ , \\ &\text{(b)}\quad |\varphi(x)| < |G_{x_0}(x)| && \text{in } [x_0,\infty)\ , \\ &\text{(c)}\quad |w(x,0)| < H_{x_0}(x) && \text{in } [0,\infty)\ . \end{aligned}$$

Then there exists $T \in (T_1,T_2)$ such that $s(T) = x_0 + 1$, $s(t) > x_0 - \lambda_1$ in $[0,T]$, and

$$\text{(3.3b)}\qquad \begin{aligned} &\text{(a)}\quad v(x,T) > G_{x_0+1}(x) && \text{in } [0,x_0 + 1)\ , \\ &\text{(b)}\quad |v(x,T)| < |G_{x_0+1}(x)| && \text{in } [x_0 + 1,\infty)\ , \\ &\text{(c)}\quad |w(x,T)| < H_{x_0+1}(x) && \text{in } [0,\infty)\ . \end{aligned}$$

Note that if $(v(x,0),w(x,0))$ satisfies (3.3a) then we can keep repeating this result to conclude that $s(t)$ is continuously moving to the right by one unit. Hence, some sort of signal is being propagated.

To complete the proof of Theorem 1.1 we prove the following:

Theorem 3.2: Assume that (v,w) is a solution of the initial-boundary value problem (3.1). The constants δ and θ obtained in Theorem 3.1 can be chosen so that if $\varphi(x)$ satisfies (1.2) with $x_0 > \theta$, and $\varepsilon \in (0,\delta)$, then there exists a constant T_0 such that for $\lambda = s(T_0)$,

$$\text{(3.4)}\qquad \begin{aligned} &\text{(a)}\quad v(x,T_0) > G_\lambda(x) && \text{in } [0,\lambda)\ , \\ &\text{(b)}\quad |v(x,T_0)| < |G_\lambda(x)| && \text{in } [\lambda,\infty)\ , \\ &\text{(c)}\quad |w(x,T_0)| < H_\lambda(x) && \text{in } [0,\infty)\ . \end{aligned}$$

This result completes the proof of Theorem 1.1 because once (3.4) is satisfied we can apply Theorem 3.1 to conclude that $\varphi(x)$ is superthreshold.

Note that we wish to obtain a lower bound for v and an upper bound for w. To obtain these bounds we use repeated applications of the following estimates.

Since $w(x,t)$ satisfies the ordinary differential equation $w_t = \varepsilon(v - \gamma w)$, it can be written explicitly in terms of v as:

$$w(x,t) = e^{-\varepsilon\gamma t}w(x,0) + \varepsilon e^{-\varepsilon\gamma t}\int_0^t e^{\varepsilon\gamma\eta}v(x,\eta)d\eta \ .$$

Assuming that $w(x,0) \leq H_{x_0}(x)$ in $\mathbb{R}^+$ it follows that

$$(3.5) \qquad |w(x,t)| \leq H_{x_0}(x) + \varepsilon\int_0^t |v(x,\eta)|d\eta \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^+ \ ,$$

and, therefore,

$$(3.6) \qquad |w(x,t)| < H_{x_0}(x) + \varepsilon Vt \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^+ \ .$$

Using these estimates we are able to control the size of w by choosing ε small. Once we have an upper bound on w, we use the comparison theorems described in Section 2 to obtain lower bounds on v.

In the estimate (3.6), $w(x,t)$ increases linearly in time. We shall see that this provides a sufficient bound for $x < s(t)$. However, we shall need that the functions v and w decay exponentially fast to zero as $x \to \infty$. This shall be proven using Proposition 2.5, and taking advantage of the fact that for $x > s(t)$, (v,w) is a solution of the linear system of equations:

$$v_t = v_{xx} - v - w$$
$$w_t = \varepsilon(v - \gamma w) \ .$$

In what follows the reader should constantly refer to Figure 3.1 which shows that the comparison functions $G_{x_0}(x)$ and $H_{x_0}(x)$ involve many constants and

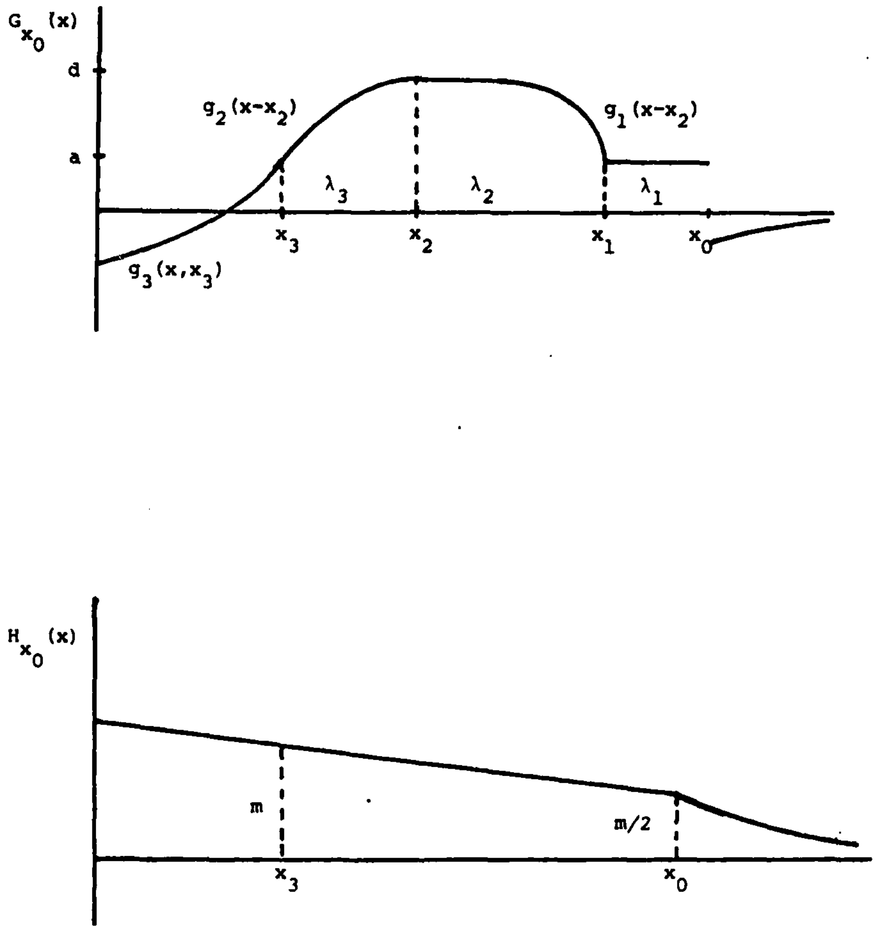

Figure 3.1. The Comparison Functions $G_{x_0}(x)$ and $H_{x_0}(x)$.

functions. We present the various properties of these constants and functions whenever they are needed in the proof of Theorem 1.1. To emphasize that the comparison functions are well defined and depend only on the parameters a and γ we also present, without motivation, their definitions in Appendix A.

We mentioned that the proof of Theorem 1.1 is split into two parts. We first prove Theorem 3.1. Until otherwise stated we assume that (3.3a) is satisfied. The proof of Theorem 3.1 consists of a number of steps. In (A) we define $G_{x_0}(x)$ and $H_{x_0}(x)$ for $x > x_0$, and show that $|v(x,T)| > G_{x_0+1}(x)$ and $|w(x,T)| < H_{x_0+1}(x)$ for $x > x_0 + 1$. In (B) we define $H_{x_0}(x)$ for $x \in [0,x_0)$, and show that $w(x,T) < H_{x_0+1}(x)$ for $x \in [0,x_0 + 1)$. In (C) we define the comparison function $G_{x_0}(x)$ for $x \in [0,x_0)$, and show that $v(x,T) > G_{x_0+1}(x)$ for $x \in (0,x_0 + 1)$. In (D) and (E) we prove there exist constants λ_1, T_1, and T_2 which depend only on the parameters a and γ such that:

(a) $s(T)$ is a well defined, continuous function satisfying

$s(T) = x_0 + 1$, for some $T \in (T_1,T_2)$,

(3.7) (b) $s(t) > x_0 - \lambda_1$ for $t \in (0,T)$. Furthermore, $v > a$ for

$x_0 - \lambda_1 < x < s(t)$, $0 < t < T$,

(c) $s(t) < x_0 + 1$ for $t < T$.

In (A), (B), and (C) we assume that (3.7) is valid. This is justified because the proof of (3.7), given in (D) and (E), does not depend on the results proven in (A), (B) or (C). The reason that the proof of Theorem 3.1 is presented in this order is to better motivate why the comparison functions $G_{x_0}(x)$ and $H_{x_0}(x)$ are defined as they are. For a more rigorous presentation of the proof of Theorem 3.1, it is suggested that the reader begin with Appendix A where the comparison functions are formally defined. He should then read (D) and (E), and conclude with (A), (B), and (C). The proof of Theorem 3.2 is given in (F).

(A) $\underline{G_{x_0}(x) \text{ and } H_{x_0}(x) \text{ for } x > x_0}$

For $x > x_0$, let $G_{x_0}(x) = -ae^{\frac{\sqrt{2}}{2}(x_0-x)}$ and $H_{x_0}(x) = \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0-x)}$ where

$$m = \frac{1}{4}\min(\frac{1}{2} - a, a) \ . \tag{3.8}$$

Recall that we are assuming that (3.7) holds. In particular, $s(T) = x_0 + 1$ for some $T > 0$, and $s(t) < x_0 + 1$ for $t < T$. Here we show that the constant δ can be chosen so that if $\varepsilon \in (0,\delta)$, then, for $x > x_0 + 1$,

$$|v(x,T)| < |G_{x_0+1}(x)| \ ,$$

and

$$|w(x,T)| < H_{x_0+1}(x) \ .$$

This is done by applying Proposition 2.5.

Let $v_0(x,t) = ae^{\frac{\sqrt{2}}{2}(x_0+1-x)}$ and $v_1(x,t) = -ae^{\frac{\sqrt{2}}{2}(x_0+1-x)}$ in $[x_0 + 1,\infty) \times [0,T]$. Note that $v(x_0 + 1,t) < a = v_0(x_0 + 1,t)$ for $t \in (0,T)$. This is because $s(t) = \sup\{x:v(x,t) = a\}$ while $T = \inf\{t:s(t) = x_0 + 1\}$. Furthermore, $v(x_0 + 1,t) > -a = v_1(x_0 + 1,t)$ for $t \in (0,T)$. This follows from the following comparison argument which shows that $v(x,t) > -a$ for $x > s(t)$, $t \in (0,T)$. Note that $v(s(t),t) = a > -a$ for $t > 0$, and $v(x,0) > G_{x_0}(x) > -a$ for $x > x_0$. In order to apply Theorem 2.1 to conclude that $v > -a$ for $x > s(t)$, $t \in (0,T)$ we must show that $Lv > L(-a)$. Note that $Lv = -w$, while $L(-a) = -a$. From (3.6) it follows that for $x > s(t)$, $t \in (0,T)$,

$$|w(x,t)| < \frac{m}{2} + \varepsilon VT < m$$

if $\varepsilon < \frac{m}{2VT}$. So one condition we must impose on δ is that $\delta < \frac{m}{2VT}$. Since $m < a$ the result follows.

Let $w_k(x,t)$, $k = 0,1$, be the solutions of the equations:

$$w_{k_t} = \varepsilon(v_k - \gamma w_k) \qquad \text{in} \quad (x_0 + 1,\infty) \times R^+ ,$$

$$w_k(x,0) = \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0+1-x)} \qquad \text{in} \quad (x_0 + 1,\infty) .$$

Note that $v_1(x,0) \leq v(x,0) \leq v_0(x,0)$ and $w_1(x,0) \leq w(x,0) \leq w_0(x,0)$ in $(x_0 + 1,\infty)$. In order to apply Proposition 2.5 to conclude that

$|v(x_0,t)| \leq ae^{\frac{\sqrt{2}}{2}(x_0+1-x)}$ and $|w(x,t)| \leq \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0+1-x)}$ in $(x_0 + 1,\infty) \times (0,T)$, we must show that

$$Lv_0 \geq -w_1$$

and

$$Lv_1 \leq -w_0 \quad \text{in} \quad (x_0 + 1,\infty) \times (0,T) .$$

Recall that L is the operator defined by $Lv \equiv v_t - v_{xx} - v$. Note that

$Lv_0 = \frac{a}{2} e^{\frac{\sqrt{2}}{2}(x_0+1-x)}$ and $Lv_1 = -\frac{a}{2} e^{\frac{\sqrt{2}}{2}(x_0+1-x)}$. On the other hand, it follows from (3.5) that, in $(x_0 + 1,\infty) \times (0,T)$,

$$\begin{aligned} |w_0(x,t)| &\leq H_{x_0}(x) + \varepsilon \int_0^T |v_0(x,\eta)|d\eta \\ &\leq \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0-x)} + \varepsilon \int_0^T ae^{\frac{\sqrt{2}}{2}(x_0+1-x)} d\eta \\ &= \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0-x)} + \varepsilon aTe^{\frac{\sqrt{2}}{2}(x_0+1-x)} \\ &= [\frac{m}{2} e^{-\frac{\sqrt{2}}{2}} + \varepsilon aT]e^{\frac{\sqrt{2}}{2}(x_0+1-x)} . \end{aligned}$$

Therefore, if ε is chosen so that

$$\varepsilon < \frac{m}{2aT} [1 - e^{-\frac{\sqrt{2}}{2}}] , \tag{3.9}$$

then

$$|w_0(x,t)| \leq \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0+1-x)} = H_{x_0+1}(x) . \tag{3.10}$$

Since $m < a$ we conclude that

$$Lv_1 \leq -w_0 \quad \text{in} \quad (x_0 + 1,\infty) \times (0,T) .$$

A similar computation shows that if (3.9) is satisfied, then $Lv_0 \geq -w_1$ in $(x_0 + 1,\infty) \times (0,T)$. We can therefore apply Proposition 2.5 to conclude that

$$|v(x,T)| \leq v_0(x,T) = G_{x_0+1}(x)$$

and

$$|w(x,T)| \leq w_0(x,T) \quad \text{for} \quad x \geq x_0 + 1 .$$

From (3.10) it follows that $|w(x,T)| \leq H_{x_0+1}(x)$ for $x \geq x_0 + 1$.

(B) $\underline{H_{x_0}}$ $\underline{\text{for } x \in (0,x_0)}$

For $x \in (0,x_0)$ set $H_{x_0}(x) = \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)} (x_0-x) + \frac{m}{2}$. The constants λ_1, λ_2, and λ_3 will be defined later, in C, when we discuss $G_{x_0}(x)$ for $x \in (0,x_0)$. For now we just assume that they are well defined positive constants which do not depend on the parameter ε. Here we show that if ε is sufficiently small, then $w(x,T) \leq H_{x_0+1}(x)$ for $x \in (0,x_0 + 1)$. In fact, we prove a stronger result which is needed later.

Consider the line $\ell(t) = x_0 + t/T$. We show that if ε is sufficiently small, then $w(x,t) \leq H_{\ell(t)}(x)$ for $(x,t) \in (0,x_0 + 1) \times (0,T)$. Since $\ell(T) = x_0 + 1$ this implies the desired result.

First suppose that $x \in (0,x_0)$. Assume that

$$\varepsilon < \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)VT} . \tag{3.11}$$

Then, from (3.6), it follows that

$$\begin{aligned}
|w(x,t)| &\leq H_{x_0}(x) + \varepsilon Vt \\
&\leq \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)} (x_0 - x) + \frac{m}{2} + \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)} \frac{t}{T} \\
&= \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)} (x_0 + t/T - x) + \frac{m}{2} \\
&= H_{\ell(t)}(x) .
\end{aligned}$$

For $x \in (x_0, x_0 + 1)$, $H_{x_0}(x) = \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0 - x)}$. Assume that

$$(3.12) \qquad \varepsilon < \frac{1}{VT_2} \min_{0 \le x \le 1} \left\{ \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)} (1 - x) + \frac{m}{2} - \frac{m}{2} e^{-\frac{\sqrt{2}}{2} x} \right\} .$$

Then, from (3.6),

$$|w(x,t)| \le H_{x_0}(x) + \varepsilon Vt$$

$$\le \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3} (x_0 + \frac{t}{T} - x) + \frac{m}{2}$$

$$= H_{\ell(t)}(x) .$$

(C) $\underline{G_{x_0}(x) \quad \text{for} \quad x \in [0, x_0)}$

We now define the comparison function $G_{x_0}(x)$ for $x \in (0, x_0)$, and show that if ε is sufficiently small, then $v(x,T) > G_{x_0+1}(x)$ for $x \in (0, x_0 + 1)$. Let

$$(3.13) \qquad G_{x_0}(x) = \begin{cases} a & \text{for} \quad x_1 \equiv x_0 - \lambda_1 < x \le x_0 \\ g_1(x - x_2) & \text{for} \quad x_2 \equiv x_1 - \lambda_2 < x \le x_1 \\ g_2(x - x_2) & \text{for} \quad x_3 \equiv x_2 - \lambda_3 < x \le x_2 \\ g_3(x, x_3) & \text{for} \quad 0 \le x \le x_3 . \end{cases}$$

The constant λ_1 is determined later in (D). It shall be chosen so that (3.7) is satisfied. The constants λ_2, λ_3 and functions $g_1(x)$, $g_2(x)$, $g_3(x, x_3)$ are defined as follows.

$\underline{\lambda_3}$, $\underline{g_2(x)}$: Recall from (3.8) that $m = \frac{1}{4} \min(\frac{1}{2} - a, a)$. Set

$$(3.14) \qquad \lambda_3 = \log\left[2 \frac{(1 - (a + m))}{m}\right] .$$

For $x \in [-\lambda_3, 0]$, define $g_2(x)$ to be the solution of the differential equation

$$g_2'' - g_2 = m - 1 ,$$

$$g_2'(0) = 0, \quad g_2(-\lambda_3) = a .$$

Note that

(3.15) $$g_2(x) = \left[\frac{a + m - 1}{e^{\lambda_3} + e^{-\lambda_3}}\right](e^x + e^{-x}) + 1 - m \qquad \text{for} \quad x \in [-\lambda_3, 0] .$$

Using this formula and (3.14), one easily verifies that

(3.16)
(a) $g_2'(x) > 0$ for $x \in [-\lambda_3, 0)$,

(b) $1 - 2m < g_2(0) < 1 - m$,

(c) $g_2'(-\lambda_3) > \frac{1}{2}$.

Let

(3.17) $$d = g_2(0) .$$

$\underline{\lambda_2}$, $\underline{g_1(x)}$: Recall, from (3.7), that we are assuming that there exist constants T_1 and T_2 such that $s(T) = x_0 + 1$ for some $T \in (T_1, T_2)$. Let

$$\bar{\lambda}_2 = \frac{2}{1 - (d + m)} \left[\frac{1 + T_1}{T_1}\right](d - a) ,$$

and

(3.18) $$\lambda_2 = \bar{\lambda}_2 + \lambda_3 .$$

Note that $\lambda_2 > \bar{\lambda}_2 > 1$. Let

(3.19) $$g_1(x) = \begin{cases} d & \text{for} \quad 0 < x < \lambda_3 \\ d - \left[\frac{d - a}{\bar{\lambda}_2^2}\right](x - \lambda_3)^2 & \text{for} \quad \lambda_3 < x < \lambda_2 . \end{cases}$$

The important properties of $g_1(x)$ are:

(3.20)
(a) $g_1(0) = d$,

(b) $g_1'(0) = 0$,

(c) $g_1(\lambda_2) = a$,

(d) $g_1'(x) < 0$ for $x \in (0, \lambda_2)$,

(e) $g_1'' + \frac{1}{T_1} g_1' > m + d - 1$ for $x \in (\lambda_3, \lambda_2)$.

Property (e) follows because for $x \in (\lambda_3, \lambda_2)$,

$$g_1'' + \frac{1}{T_1} g_1' = -2\left[\frac{d-a}{\bar{\lambda}_2^2}\right]\left[1 + \frac{(x-\lambda_3)}{T_1}\right] > -2\left[\frac{d-a}{\bar{\lambda}_2}\right]\left[\frac{T_1 + (x-\lambda_3)}{T_1}\right]$$

$$> -2\left[\frac{d-a}{\bar{\lambda}_2}\right]\left[\frac{1+T_1}{T_1}\right] = m + d - 1 .$$

<u>$g_3(x,x_3)$</u>: For $x \in [0,x_3]$, define $g_3(x,x_3)$ to be the solution of the differential equation:

$$g_3'' - g_3 = -\frac{m}{2}(x - x_3) + m ,$$

$$g_3'(0,x_3) = 0, \quad g_3(x_3,x_3) = a .$$

Note that

$$g_3(x,x_3) = \left[\frac{a + m - \frac{m}{2} e^{-x_3}}{e^{x_3} + e^{-x_3}}\right][e^x + e^{-x}] + \frac{m}{2} e^{-x} + \frac{m}{2}(x - x_3) - m . \tag{3.21}$$

The following properties follow from this formula.

(3.22)

(a) $\frac{d}{dx} g_3(x,x_3)|_{x=x_3} < \frac{1}{2}$

(b) Choose λ_4 so that $e^{\lambda_4} - e^{-\lambda_4} > 1$ and $\delta > 0$. For $t > 0$ let $\ell_1(t) = \lambda_4 + \delta t$. Then, $\frac{\partial}{\partial t} g_3(x,\ell(t)) < 0$ for $x \in [0,\ell_1(t)]$, $t \in R^+$. This is because,

$$\frac{\partial}{\partial t} g_3(x,\ell(t)) = -\delta\left[\frac{a(e^{\ell(t)} + e^{-\ell(t)}) + m(e^{\ell(t)} - e^{-\ell(t)} - 1)}{(e^{\ell(t)} + e^{-\ell(t)})^2}\right][e^x + e^{-x}] - \frac{\delta m}{2}$$

$$< -\delta\left[\frac{a(e^{\ell(t)} + e^{-\ell(t)}) + m(e^{\lambda_4} - e^{-\lambda_4} - 1)}{(e^{\ell(t)} + e^{-\ell(t)})^2}\right][e^x + e^{-x}] - \frac{\delta m}{2}$$

$$< 0 .$$

We now show that if ε is sufficiently small, then $v(x,T) > G_{x_0+1}(x)$ for $x \in (0,x_0 + 1)$. We actually prove a little more. For $t \in [0,T]$, let $\ell(t) = x_0 + \frac{t}{T}$ and $\gamma(t) = \inf\{\ell(t),s(t)\}$. We show using the comparison theorems that $G_{\ell(t)}(x) < v(x,t)$ for $(x,t) \in G = \{(x,t): x \in [0,\gamma(t)),\ t \in (0,T)\}$. Since $\gamma(T) = s(T)$ this certainly implies the desired result.

The proof is by contradiction. Suppose there exists some point $p = (\bar{x},\bar{t}) \in G$ such that $G_{\ell(\bar{t})}(\bar{x}) = v(\bar{x},\bar{t})$. From (3.7b) it follows that p can be chosen so that $G_{\gamma(t)}(x) < v(x,t)$ for $(x,t) \in G,\ t < \bar{t}$.

In order to use a comparison argument to obtain a contradiction it is necessary to show that $Lv > LG_{\ell(t)}(x)$ in G. To estimate Lv we use the results of (B) where it was shown that $w(x,t) \le H_{\ell(t)}(x)$ for $x \in (0,\ell(t)),\ t \in (0,T)$. Therefore,

$$w(x,t) < H_{\ell(t)}(x) = \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)}\left(x_3 + \frac{t}{T} - x\right) + m$$

$$(3.23)\qquad < \begin{cases} -\frac{m}{2}\left(x - \left(x_3 + \frac{t}{T}\right)\right) + m & \text{for } 0 < x < x_3 + \frac{t}{T}, \\ \\ m & \text{for } x_3 + \frac{t}{T} \le x \le \ell(t). \end{cases}$$

Here we used that $\lambda_1 + \lambda_2 + \lambda_3 > 1$. This is justified because $\lambda_2 > 1$ (see the remarks following (3.18)).

We first show that it is impossible for $\bar{x} \in (0,x_3 + \frac{\bar{t}}{T})$. If this were the case, then, since $G_{\ell(t)}(x) < a$ for $x \in (0,x_3 + \frac{\bar{t}}{T})$, there exists a constant β such that $v(x,t) < a$ and $G_{\ell(t)}(x) < a$ in the rectangle $R = (\bar{x} - \beta,\bar{x} + \beta) \times (\bar{t} - \beta,\bar{t})$. However, $G_{\ell(t)}(x) < v(x,t)$ on the left, right, and bottom sides of R. We show that $LG_{\ell(t)}(x) < Lv$ in R so that we can apply Theorem 2.1 to obtain the desired contradiction. Note that, in R, $G_{\ell(t)}(x) = g_3(x - (x_3 + \frac{t}{T}),x_3 + \frac{t}{T})$, and therefore,

$$LG_{\ell(t)}(x) = \frac{\partial}{\partial t} g_3\left(x - \left(x_3 + \frac{t}{T}\right),x_3 + \frac{t}{T}\right) - \frac{\partial^2}{\partial x^2} g_3\left(x - \left(x_3 + \frac{t}{T}\right),x_3 + \frac{t}{T}\right)$$
$$+ g_3\left(x - \left(x_3 + \frac{t}{T}\right),x_3 + \frac{t}{T}\right).$$

From (3.22b) it follows that

$$LG_{\ell(t)}(x) < -\frac{\partial^2}{\partial x^2} g_3(x - (x_3 + \frac{t}{T}), x_3 + \frac{t}{T}) + g_3(x - (x_3 + \frac{t}{T}), x_e + \frac{t}{T})$$

$$= \frac{m}{2}(x - (x_3 + \frac{t}{T})) - m .$$

On the other hand, in R, $v < a$ and, therefore, $Lv = -w$. From (3.23) it follows that $LG_{\ell(t)}(x) < Lv$ in R.

A similar argument shows that it is impossible for $\bar{x} \in (x_3 + \frac{\bar{t}}{T}, x_2 + \frac{\bar{t}}{T})$ or $\bar{x} \in (x_2 + \frac{\bar{t}}{T}, x_1 + \frac{\bar{t}}{T})$. If this were the case, then, as before, there exists a constant β such that $G_{\ell(t)}(x) > a$ and $v(x,t) > a$ in the rectangle R defined by $R = (\bar{x} - \beta, \bar{x} + \beta) \times (\bar{t} - \beta, \bar{t})$. Then $G_{\ell(t)}(x) < v(x,t)$ on the left, right, and bottom sides of R. We show that $LG_{\ell(t)}(x) < Lv$ in R so that we can apply Theorem 2.1 to obtain the desired contradiction.

First suppose that $\bar{x} \in (x_3 + \frac{\bar{t}}{T}, x_2 + \frac{\bar{t}}{T})$. Then $G_{\ell(t)}(x) = g_2(x - (x_3 + \frac{\bar{t}}{T}))$. From (3.16a) it follows that

$$LG_{\ell(t)}(x) = -\frac{1}{T} g_2'(x - (x_3 + \frac{t}{T})) - g_2''(x - (x_3 + \frac{t}{T})) + g_2(x - (x_3 + \frac{t}{T}))$$

$$< -g_2''(x - (x_3 + \frac{t}{T})) + g_2(x - (x_3 + \frac{t}{T}))$$

$$= 1 - m .$$

On the other hand, from (3.23), $w(x,t) < m$ in R. Therefore, $Lv = 1 - w > 1 - m > LG_{\ell(t)}(x)$ in R.

If $\bar{x} \in (x_2 + \frac{\bar{t}}{T}, x_1 + \frac{\bar{t}}{T})$, then $G_{\ell(t)}(x) = g_1(x - (x_2 + \frac{t}{T}))$ in R. From (3.20) it follows that

$$LG_{\ell(t)}(x) = -\frac{1}{T} g_1'(x - (x_2 + \frac{t}{T})) - g_1''(x - (x_2 + \frac{t}{T})) + g_1(x - (x_2 + \frac{t}{T}))$$

$$< -\frac{1}{T_1} g_1'(x - (x_2 + \frac{t}{T})) - g_1''(x - (x_2 + \frac{t}{T})) + g_1(x - (x_2 + \frac{t}{T}))$$

$$< 1 - (m + d) + d$$

$$= 1 - m .$$

However, from 3.23, $w(x,t) < m$ in R. Therefore, $Lv = 1 - w > 1 - m > LG_{\ell(t)}(x)$ in R.

Similarly $\bar{x} \neq 0$. In this case we can argue as before chosing a rectangle of the form $R = (0,\beta) \times (\bar{t} - \beta, \bar{t})$.

It remains to show that it is impossible for $\bar{x} = x_k + \frac{t}{T}$, for $k = 1, 2,$ or 3. From Theorem 2.2 it follows that this is impossible if

$$\frac{\partial}{\partial x} G_{\ell(\bar{t})}((x_k + \frac{\bar{t}}{T})^-) < \frac{\partial}{\partial x} G_{\ell(\bar{t})}((x_k + \frac{\bar{t}}{T})^+) \quad \text{for} \quad k = 1,2,3 .$$

Here $g'(x^-)$ denotes the left sided derivative of g at x, while $g'(x^+)$ denotes the right sided derivative.

If $\bar{x} = x_3 + \frac{\bar{t}}{T}$, then, from (3.16c) and (3.22a),

$$\frac{\partial}{\partial x} G_{\ell(\bar{t})}(\bar{x}^-) = \frac{\partial}{\partial x} g_3(\bar{x},\bar{x}) < \frac{1}{2} ,$$

while,

$$\frac{\partial}{\partial x} G_{\ell(\bar{t})}(\bar{x}^+) = g_2'(-\lambda_3) > \frac{1}{2} .$$

If $\bar{x} = x_2 + \frac{\bar{t}}{T}$, then,

$$\frac{\partial}{\partial x} G_{\ell(\bar{t})}(\bar{x}^-) = g_2'(0) = 0 ,$$

while,

$$\frac{\partial}{\partial x} G_{\ell(\bar{t})}(\bar{x}^+) = g_1'(0) = 0 .$$

Finally, if $\bar{x} = x_1 + \frac{\bar{t}}{T}$, then, from (3.20d),

$$\frac{\partial}{\partial x} G_{\ell(\bar{t})}(\bar{x}^-) = g_1'(\lambda_2) < 0 ,$$

while $G_{\ell(\bar{t})}(x) = a$ for $x \in (x_1 + \frac{\bar{t}}{T}, x_0 + \frac{\bar{t}}{T})$ and, therefore,

$$\frac{\partial}{\partial x} G_{\ell(\bar{t})}(\bar{x}^+) = 0 \ .$$

(D) The Curve $s(t)$ - A Formal Presentation

We now show that $s(t)$ is a well defined continuous function satisfying $s(T) = x_0 + 1$ for some time T. We also verify (3.7). The proof involves comparing $v(x,t)$ to solutions of scalar equations. In order to motivate the comparison functions we first present a formal proof that $s(T) = x_0 + 1$ for some time T. In this formal proof we assume that $s(t)$ is a well defined, continuous function, and the comparison theorems given in Section 2 are valid when the operator L is replaced by the operator L_1 defined by

$$L_1 u \equiv u_t - u_{xx} - f(u) \ .$$

Note that since f is discontinuous the comparison theorems are not really valid. By making these assumptions, however, we are able to present the main ideas of the proof while avoiding the difficulties due to the discontinuity of f. After the formal proof we present a rigorous, analytic proof in (E).

From Theorem 2.4 we conclude that there exist constants C_1 and T_2 which depend only on the parameter a and have the following properties.

Suppose that $\lambda_1 > C_1$, and $u(x,t)$ is the solution of the equation:

$$(3.24) \qquad \begin{aligned} u_t &= u_{xx} + f(u) - 2m \quad \text{in} \quad (x_2, \infty) \times R^+ \ , \\ u_x(x_2, t) &= 0 \quad \text{in} \quad R^+ \ . \end{aligned}$$

Recall that x_2 was defined in (3.13) as $x_2 = x_0 - (\lambda_1 + \lambda_2)$. For initial conditions we take $u(x,0) = \psi(x)$ where, for now, $\psi(x)$ is any function, defined for $x > x_2$, which satisfies

$$
(3.25)\qquad
\begin{array}{lll}
\text{(i)} & \psi(x) \in (-a, 1-2m) & \text{for } x > x_2\ , \\
\text{(ii)} & \psi(x) > a & \text{for } x \in (x_2, x_0 - 1)\ , \\
\text{(iii)} & \psi(x_0 - 1) = a\ , & \\
\text{(iv)} & \psi(x) \in C^2((x_2,\infty))\ , & \\
\text{(v)} & \psi'(x) < 0 & \text{for } x > x_2\ .
\end{array}
$$

Then, the curve $\sigma(t)$ given by $\sigma(t) = \sup\{x : u(x,t) = a\}$ is a well defined function. Furthermore,

$$
(3.26)\qquad
\begin{array}{l}
\sigma(t) \in C^1(R^+), \quad \sigma(t) > x_1 + 1 \ \text{ in } \ \mathbb{R}^+\ , \\
\lim_{t\to\infty} \sigma(t) = +\infty, \ \text{ and } \ \sigma(T_2) > x_0 + 1\ .
\end{array}
$$

We show, using a comparison argument, that $v(x,t) > u(x,t)$ in the set $G = \{(x,t) : x > x_2,\ t \in (0,T_2)\}$. This will imply that $s(t) > \sigma(t)$ in $(0,T_2)$, and therefore, $s(T_2) > x_0 + 1$. We then let $T = \inf\{t : s(T) = a\}$.

In order to apply the comparison theorems to conclude that $v > u$ in G we must show that:

$$
(3.27)\qquad
\begin{array}{lll}
\text{(a)} & v(x,0) > u(x,0) & \text{for } x > x_2\ , \\
\text{(b)} & L_1 v > L_1 u & \text{in } G\ , \\
\text{(c)} & v(x_2,t) > u(x_2,t) & \text{for } t \in (0,T_2)\ .
\end{array}
$$

Note that since $v(x,0) = \varphi(x)$ satisfies (3.3a), one can certainly find a function $\psi(x)$ which satisfies (3.25) and (3.27a).

Since $L_1 v = -w$ and $L_1 u = -2m$, proving (3.27b) is equivalent to showing that $w(x,t) < 2m$ in G. This was proven in (3.23).

It remains to verify (3.27c). We show that $u(x_2,t) < 1 - 2m < v(x_2,t)$ for $t \in (0,T_2)$. The first inequality follows from a simple application of Theorem 2.3 which shows that $u(x,t) < 1 - 2m$ in G. This is because $u(x,0) < 1 - 2m$ for $x > x_2$, $u_x(x_2,t) = 0$, and $L_1 u = -2m = L_1(1 - 2m)$ in G.

To complete our formal proof that $s(T) = x_0 + 1$ for some T it remains to show that $v(x_2,t) > 1 - 2m$ for $t \in (0,T_2)$. This is done by showing, via a comparison argument, that $v(x,t) > h(x)$ in $\mathbb{R}^+ \times (0,T_2)$ for some continuous function $h(x)$ satisfying $h(x_2) > 1 - 2m$. The comparison function $h(x)$ is defined as follows.

Let $g(x)$ be the solution of the equations:

$$g'' - g = m \quad \text{in } \mathbb{R}^+$$
$$g(0) = a, \quad g'(0) = \frac{1}{2} .$$

Note that

(3.28) $$g(x) = [\frac{a + m - 1/2}{2}][e^x - e^{-x}] + [a + m]e^{-x} - m .$$

One easily verifies that $g'(x) < 0$ in $\mathbb{R}^+$ and $\lim_{x \to \infty} g(x) = -\infty$. Define C_2 by $g(C_2) = -a$. Note that C_2 depends only on the parameter a. Assume that

(3.29) $$\lambda_1 > C_2 .$$

Define $h(x)$ by

(3.30) $$h(x) = \begin{cases} G_{x_0}(x) & \text{for } 0 \leq x < x_2 \\ g_2(x_2 - x) & \text{for } x_2 \leq x \leq x_2 + \lambda_3 \\ g(x - (x_2 + \lambda_3)) & \text{for } x_2 + \lambda_3 < x . \end{cases}$$

The important properties of $h(x)$ are:

(3.31)
(a) $h(x) \leq G_{x_0}$ in $\mathbb{R}^+$,

(b) $h(x_2) = G_{x_0}(x_2) > 1 - 2m$,

(c) $L_1 h < L_1 v$ in G wherever $L_1 h$ is defined,

(d) $h(x)$ is a smooth function except at $x = x_3$ and $x = x_2 + \lambda_3$ where $h'(x^-) < h'(x^+)$.

(3.31a) is true because

$$h(x) = \begin{cases} G_{x_0}(x) & \text{for } 0 \leq x < x_2 , \\ g_2(x_2 - x) < d = G_{x_0}(x) & \text{for } x_2 \leq x < x_2 + \lambda_3 , \\ g(x - (x_2 + \lambda_3)) < a \leq G_{x_0}(x) & \text{for } x_2 + \lambda_3 \leq x < x_0 , \\ g(x - (x_2 + \lambda_3)) < -a < G_{x_0}(x) & \text{for } x_0 \leq x . \end{cases}$$

(3.31b) is true because of (3.16b). (3.31c) is true because

$$L_1 h = \begin{cases} \frac{m}{2}(x - x_3) - m & \text{for } x \in (0,x_3) \\ -m & \text{for } x > x_3 , \end{cases}$$

while $L_1 v = -w$, where $w(x,t)$ satisfies (3.23). Finally, (3.31d) is true because $h'(x_3^-) = g_3'(x_3,x_3) < \frac{1}{2}$ (see (3.22a)), while $h'(x_3^+) = g_2'(-\lambda_3) > \frac{1}{2}$ (see (3.16c)). On the other hand, $h'((x_2 + \lambda_3)^-) = -g_2'(-\lambda_3) < -\frac{1}{2}$, while $h'((x_2 + \lambda_3)^+) = g'(0) = -\frac{1}{2}$.

We now prove that $v(x,t) > h(x)$ in $\mathbb{R}^+ \times (0,T_2)$. If this were not the case then, since $h(x) < v(x,0)$ in $\mathbb{R}^+$ and $\lim_{x\to\infty} h(x) = -\infty$, there must exist some point $(\bar{x},\bar{t})$ such that $h(\bar{x}) = v(\bar{x},\bar{t})$ and $h(x) < v(x,t)$ for $t < \bar{t}$. From (3.31d) it follows that $\bar{x} \neq x_3$ and $\bar{x} \neq x_2 + \lambda_3$. Now suppose that $\bar{x} \neq 0$ and $h(\bar{x}) = v(\bar{x},\bar{t}) < a$ ($> a$). Then there must exist a positive constant β such that $h < a$ and $v < a$ ($> a$) in the rectangle $R = (\bar{x} - \beta, \bar{x} + \beta) \times (\bar{t} - \beta, \bar{t})$. However, $h < v$ on the bottom, left, and right hand sides of R. Since in R, $h(x)$ and $v(x,t)$ are solutions of linear differential equations, we conclude from (3.31c) and Theorem 2.1 that this is impossible. A similar argument shows that it is impossible for $\bar{x} = 0$.

(E) The Curve $s(t)$ - A Rigorous Presentation

We now give a rigorous proof that $s(t)$ is a well defined continuous function such that $s(T) = x_0 + 1$ for some time T. We also show that (3.7) holds. The proof is broken up into a few lemmas. In what follows we shall use the constant T_2 which was defined in (D).

Lemma 3.3: Let $\mathcal{D} = (x_2,\infty) \times (0,T_2)$, $I_t = \{x > x_2 : v(x,t) = a\}$ and $I = \{(x,t) \in \mathcal{D} : x \in I_t\}$. Then I_t is nonempty for each t and, therefore, $s(t)$ is a well defined function.

Proof: Recall the function $h(x)$ defined in (D). A rigorous proof was given in (D) that $v(x,t) > h(x)$ in $\mathcal{D}$. From this it follows that $v(x_2,t) > h(x_2) = d$. Furthermore, since $h(x_2 + \lambda_3) = a$ it follows that $s(t) > x_2 + \lambda_3$ whenever $s(t)$ is defined.

We now wish to find an upper bound in $v(x,t)$ and $s(t)$. This is done using a comparison argument. Recall that $w(x,t) < m$ in $\mathcal{D}$. This was proven in (3.23). It is therefore natural to choose a comparison function, $z(x,t)$, which is a solution of the differential equation

$$(3.32)\qquad \begin{aligned} &z_t = z_{xx} + f(z) + m \quad \text{in } \mathcal{D}\,, \\ &z_x(x_2,t) = 0 \qquad\qquad \text{for } t > 0\,. \end{aligned}$$

We must choose $z(x,0)$ so that $z(x,0) > v(x,0)$ for $x > x_2$. Since $|v(x,0)| < V$ for $x \in [x_2,x_0]$ and $|v(x,0)| < |G_{x_0}(x)| = ae^{\frac{\sqrt{2}}{2}(x_0-x)}$ for $x > x_0$, we let $z(x,0) = \xi(x)$ where $\xi(x)$ in a smooth function which satisfies

$$(3.33)\qquad \begin{aligned} &\text{(a)}\quad \xi(x) \in (V,2V) && \text{for } x_2 \le x \le x_0\,, \\ &\text{(b)}\quad \xi'(x) < 0 && \text{for } x > x_2\,, \\ &\text{(c)}\quad \xi(x) = ae^{\frac{\sqrt{2}}{2}(x_0+1/2-x)} && \text{for } x > x_0 + \tfrac{1}{2}\,, \\ &\text{(d)}\quad \xi(x) \in C^2(x_2,\infty)\,. \end{aligned}$$

From Theorem 2.4 it follows that there exists a constant C_0 such that if $x_0 - x_2 > C_0$, then the curve $\sigma(t)$, given by $z(\sigma(t);t) = a$, $\sigma(0) = x_0 + \frac{1}{2}$ is a well defined, smooth function which satisfies $\lim_{t\to\infty} \sigma(t) = \infty$. In order to guarantee that $x_0 - x_2 > C_0$ we assume that $\lambda_1 > C_0$. Note that the constant C_0 depends only on the parameters a and γ. We now show that $v(x,t) < z(x,t)$ in $\mathcal{D}$. This will complete the proof of the lemma. It will also follow that $s(t) < \sigma(t)$ in $(0,T_2)$.

Suppose it were not true that $v < z$ in $\mathcal{D}$. Let $t_1 = \inf\{t : v(x,t) \ge z(x,t)$ for some $x \ge x_2\}$. First suppose that there exists $y > x_2$ such that $v(y,t_1) = z(y,t_1)$ and $v(x,t) < z(x,t)$ in $[x_2,\infty) \times (0,t_1)$. Since $v(x,0) < z(x,0)$ it follows that $t_1 > 0$. If $v(y,t_1) < a$ there must be a rectangle $R = (y-\beta, y+\beta) \times (t_1-\beta, t_1)$ such that $v < a$ and $z < a$ in R. Since $v < z$ on the left, right, and bottom sides of R, and $Lv = -w < m = Lz$ in R we obtain, from Theorem 2.1, the desired contradiction. A similar argument shows that it is impossible for $v(y,t_1) > a$.

Now suppose that $v(y,t_1) = z(y,t_1) = a$; that is, $y = \sigma(t_1)$. Since $v(x,t) < z(x,t)$ for $t < t_1$, it follows that $s(t) < \sigma(t)$ for $t < t_1$. Therefore, $v < a$

for $x > \sigma(t)$, $t \in (0,t_1)$. In this region, v and z are both solutions of linear differential equations with $Lv < Lz$. It now follows from Theorem 2.3 that $v_x(y,t_1) < z_x(y,t_1)$. Since v and z are both continuously differentiable functions this implies that $v(x,t_1) > z(x,t_1)$ for some $x < x_1$. This is a contradiction.

Finally suppose there exists a sequence of points (y_k,t_k) such that $v(y_k,t_k) > z(y_k,t_k)$ for each k, $t_k \downarrow t_1$, and $y_k \to \infty$ as $k \to \infty$. Since $v(\sigma(t),t) < z(\sigma(t),t) = a$ for $t \in (0,t_1)$ there must exist a positive constant β such that $v(\sigma(t),t) < z(\sigma(t),t) = a$ for $t \in (0,t_1+\beta)$. We also have that $v(x,0) > z(x,0)$ for $x > \sigma(0)$, and $Lv < Lz$ for $x > \sigma(t)$, $t \in (0,t_1+\beta)$. From Theorem 2.1 it now follows that $v < z$ for $x > \sigma(t)$, $t \in (0,t_1+\beta)$. This contradiction completes the proof of the Lemma.

<u>Lemma 3.4</u>: There exists a constant T_1, which depends only on the parameters a and γ, such that $s(t) < x_0 + 1$ for $t \in (0,T_1)$.

<u>Proof</u>: Recall the functions $z(x,t)$ and $\sigma(t)$ defined in the previous lemma. It was shown that $v(x,t) < z(x,t)$ in $\mathcal{D}$ and $s(t) < \sigma(t)$ in $(0,T_2)$. We show that there exists a constant T_1 such that $\sigma(t) < x_0 + 1$ for $t \in (0,T_1)$. Note that $z(x,t)$ and $\sigma(t)$ depend on x_0 as well as the parameters a and γ. We must choose T_1 so that it depends only on the parameters a and γ. To do this we construct the following comparison function.

Let $z_1(x,t)$ be the solution of the differential equation

$$\begin{aligned} z_{1t} &= z_{1xx} + f(z_1) + m \quad \text{in } \mathbb{R}\times\mathbb{R}^+ , \\ z_1(x,0) &= \xi_1(x) \quad \text{in } \mathbb{R} , \end{aligned} \tag{3.34}$$

where $\xi_1(x)$ satisfies:

$$\begin{aligned} &(a)\quad \xi_1(x) \in C^2(\mathbb{R}) , \\ &(b)\quad \xi_1(x) > 2V \quad \text{for } x \in (-\infty,\tfrac{1}{2}) , \\ &(c)\quad \xi_1'(x) < 0 \quad \text{for } x \in \mathbb{R} \\ &(d)\quad \xi_1(x) = ae^{\frac{\sqrt{2}}{2}\left(\frac{3}{4}-x\right)} \quad \text{for } \tfrac{3}{4} < x < \infty . \end{aligned} \tag{3.35}$$

Define $\sigma_1(t)$ by $z_1(\sigma_1(t),t) = a$, $\sigma_1(0) = x_0$. We would like to apply Theorem 2.4 to conclude that $\sigma_1(t)$ is a well defined, smooth function such that $\lim_{t\to\infty} \sigma_1(t) = \infty$. However, $z_1(x,0) \neq z_1(-x,0)$, and, therefore, the assumption (1.2c) is not satisfied. Instead, we are assuming that $z_{1x}(x,0) < 0$ in $\mathbb{R}$. One finds, however, that proof of Theorem 2.4 is easier with this assumption. The reason being that if $z_{1x}(x,0) < 0$ then there is a unique curve, $\sigma_1(t)$, such that $z_1(\sigma_1(t),t) = a$, while if (1.2c) holds then there are two curves, $\sigma_1(t)$ and $-\sigma_1(t)$, such that $z_1(\sigma_1(t),t) = a$. We assume, therefore, that $\sigma_1(t)$ is well defined, and $\lim_{t\to\infty} \sigma_1(t) = \infty$. Let $T_1 = \inf\{t:\sigma_1(t) = 1\}$. Note that T_1 depends only on the parameters a and γ.

We now prove that $\sigma(t) < \sigma_1(t) + x_0$ for $t \in (0,T_1)$. This is done by showing that $z(x,t) < z_1(x - x_0,t)$ in $\mathcal{D}$. Set $z_2(x,t) = z_1(x - x_0,t)$. As usual we wish to use a comparison argument. Note, however, that $z(x,t)$ is defined only for $x > x_2$, while $z_2(x,t)$ is defined in $\mathbb{R} \times \mathbb{R}^+$. We, therefore, set

$$z_3(x,t) = \begin{cases} z(2x_2 - x,t) & \text{for } x < x_2 \\ z(x,t) & \text{for } x > x_2 , \end{cases}$$

and show that $z_3 < z_2$ in $\mathbb{R} \times (0,T_1)$. Note that since $z_x(x_2,t) = 0$, it follows that $z_3(x,t)$ is a smooth function.

Recall that L_1 is the operator defined by $L_1 v \equiv v_t - v_{xx} - f(v)$. Note that $L_1 v_2 = m = L_1 v_3$. From the definitions it follows that $z_3(x,0) < z_2(x,0)$. We now apply Theorem 4.2 of [22] to conclude that $z_3(x,t) < z_2(x,t)$ in $\mathbb{R} \times (0,T_1)$. Therefore, $z(x,t) < z_1(x - x_0,t)$ in $(x_2,\infty) \times (0,T_1)$, from which it follows that $\sigma(t) < \sigma_1(t) + x_0 < x_0 + 1$ for $t \in (0,T_1)$. Since $s(t) < \sigma(t)$ the proof of the lemma is now complete.

Lemma 3.5: $I = \{(x,t) \in \mathcal{D} | x = s(t)\}$.

Note that this lemma implies that $s(t)$ is a continuous function in $[0,T_2]$.

Proof: Suppose the lemma is not true and let $t_1 = \inf\{t: I_t \neq \{s(t)\}\}$. From the results of [23] it follows that $t_1 > 0$. Note that $s(t)$ must be continuous on

$(0,t_1)$, and, because of assumption (3.2), $\lambda \equiv \lim_{t \uparrow t_1} s(t)$ exists. We first show that $I_{t_1} = \{\lambda\}$. The proof is by contradiction.

Suppose that $v(y,t_1) = a$ for some $y \in (x_2,\lambda)$. Because of assumption (3.2), $v > a$ in some rectangle $R = (y - \beta, y + \beta) \times (t_1 - \beta, t_1)$. Since $w < m$ in R (see (3.23)) it follows that $Lv = 1 - m > a = La$ in R. Theorem 2.1 now yields the desired contradiction. A similar argument shows that it is impossible for $v(y,t_1) = a$ for some $y > \lambda$. Hence, $I_{t_1} = \{\lambda\}$.

Now suppose there exist a sequence of points $\{(y_k,t_k)\}$ and $\{(z_k,t_k)\}$, $k \geq 2$, such that:

(a) $y_k < z_k$ for $k = 2,3,\ldots$,

(b) $y_k \to \lambda$, $z_k \to \lambda$, and $t_k \downarrow t_1$ as $k \to \infty$,

(c) $v(y_k,t_k) = v(z_k,t_k) = a$ for $k = 2,3,\ldots$.

Since $v_x(x,t)$ is assumed to be a continuous function it follows that $v_x(\lambda,t_1) = 0$. We show that this is impossible.

From assumption (3.2) it follows that there exist positive constants M and δ such that for $t_1 - \delta < t < t_1$, either

(a) $s(t) < -M(t - t_1) + \lambda$,

or (b) $s(t) > M(t - t_1) + \lambda$.

First suppose that (3.2b) holds. Then there exists a constant $\delta_1 > 0$ such that $v > a$ in the trapezoid $T = \{(x,t): \lambda - \delta_1 < x < M(t - t_1) + \lambda,\ t_1 - \frac{\delta_1}{2M} < t < t_1\}$. In T, $v(x,t)$ is the solution of the linear equation $Lv = 1 - w$. Recall from (3.23) that $w < m$ in D. Therefore, in T, $Lv = 1 - w > 1 - m > a \geq L(a)$. Since $v > a$ on the left, right, and bottom sides of T, it follows from Theorem 2.2 that $v_x(\lambda,t_1) < 0$. This contradicts our previous conclusion that $v_x(\lambda,t_1) = 0$. A similar argument shows that (3.2a) leads to a contradiction.

The following lemma completes the proof of Theorem 3.1.

Lemma 3.6. $s(T) = x_0 + 1$ for some $T \in (T_1,T_2)$. Furthermore, $s(t) > x_1 + 1$ in $(0,T)$.

<u>Proof</u>: Let $\psi(x)$ be some function which satisfies (3.25), and let $u(x,t)$ be as in (D). That is u satisfies the equation (3.24) in $\mathcal{D}$ with initial conditions $u(x,0) = \psi(x)$. In (D) we gave a formal proof that $u < v$ in (D). An argument very similar to that given in Lemma 3.4 that $v < z_1$ in $\mathcal{D}$ gives a rigorous proof that $u < v$ in $\mathcal{D}$. See [24, Lemma 4.5] for details. Hence $s(t) > \sigma(t)$ in $(0,T_2)$ where $\sigma(t) = \sup\{x: u(x,t) = a\}$. Recall, from (3.26), that $\sigma(T_2) > x_0 + 1$ and $\sigma(t) > x_1 + 1$ in $(0,T_2)$. We define T by $T = \inf\{t: s(t) = x_0 + 1\}$.

(F) <u>Proof of Theorem 3.2</u>

We now prove Theorem 3.2. This will complete the proof of Theorem 1.1. Here we assume that the initial datum, $\varphi(x)$, satisfies (1.2), and $w(x,0) \equiv 0$. As usual, the proof consists of applications of the comparison theorems. We shall need both an upper and a lower bound for $v(x,t)$. The comparison functions are now described.

Let $z(x,t)$ be the solution of the differential equation:

$$z_t = z_{xx} + f(z) + m \quad \text{in } \mathbb{R} \times \mathbb{R}^+ ,$$
$$z(x,0) = \xi(x) \quad \text{in } \mathbb{R} ,$$

where $\xi(x)$ is a smooth function which satisfies:

(a) $\xi(x) \in C^2(\mathbb{R})$,

(b) $\xi(x) = V$ for $-\infty < x < 1/2$,

(c) $\xi'(x) < 0$ for $1/2 < x < \infty$,

(d) $\xi(x) = ae^{\frac{\sqrt{2}}{2}\left(\frac{3}{4} - x\right)}$ for $\frac{3}{4} < x < \infty$.

Note that a similar a function was used in step (E). Our remarks there demonstrate that the curve $\sigma(t)$, given by $z(\sigma(t),t) = a$, is a well defined, smooth function such that $\lim_{t\to\infty} \sigma(t) = \infty$. The function $z(x,t)$ will be used as an upper bound for $v(x,t)$.

For a lower bound we consider the function $u(x,t)$ defined to be the solution of the differential equation:

$$(3.36)\qquad \begin{aligned} &u_t = u_{xx} + f(u) - m && \text{in } \mathbb{R}^+ \times \mathbb{R}^+ , \\ &u_x(0,t) = 0 && \text{in } \mathbb{R}^+ , \end{aligned}$$

with initial datum, $u(x,0) = \psi(x)$, to be determined. From Theorem 2.4 there exist constants θ_1, β, and T_3 such that if $\psi(x)$ satisfies (1.2) with $x_0 > \theta_1$ then the curve $\sigma_1(t)$, given by $u(\sigma_1(t),t) = a$, is a well defined, smooth function such that $\lim_{t\to\infty} \sigma_1(t) = \infty$, $\sigma_1(t) > x_0 - \beta$ in $\mathbb{R}^+$, and $u(x,t) > d$ for $x < x_0 - \beta$, $t > T_3$. Recall that $d = g_2(0)$ was defined in (3.17).

We let $\beta_1 = \sup_{0<t<T_3} \sigma(t)$ and $C_3 = \beta + \beta_1$. From Theorem 2.4 there exists a constant T_4 such that $\sigma_1(t) > x_0 + \beta_1$ for $t > T_4$. We assume that

$$(3.37)\qquad \lambda_1 > C_3 .$$

We first show that $v(x,t) < z_1(x,t) \equiv z(x - x_0,t)$ in $\mathbb{R}^+ \times (0,T_4)$. This is done using a comparison argument. Note that v is defined in $\mathbb{R}^+ \times \mathbb{R}^+$ while z_1 is defined in $\mathbb{R} \times \mathbb{R}^+$. Therefore, we let

$$v_1(x,t) = \begin{cases} v(-x,t) & \text{for } x < 0 , \\ v(x,t) & \text{for } x \geq 0 , \end{cases}$$

and

$$w_1(x,t) = \begin{cases} w(-x,t) & \text{for } x < 0 , \\ w(x,t) & \text{for } x \geq 0 , \end{cases}$$

and show that $v_1 < z_1$ in $\mathbb{R} \times (0,T_4)$. We first prove that if ε is chosen sufficiently small, then $L_1 v_1 < L_1 z_1$ in $\mathbb{R} \times (0,T_4)$. Recall that $L_1 u \equiv u_t - u_{xx} - f(u)$. Note that $L_1 z_1 = m$. From (3.6) it follows that in $\mathbb{R} \times (0,T_4)$, $w_1 \leq \varepsilon V T_4$. Therefore, if

$$\varepsilon < \frac{m}{VT_4}$$

it follows that $L_1 v_1 < L_1 z_1$ in $\mathbb{R} \times (0,T_4)$. It is also clear that $v_1(x,0) < z_1(x,0)$. An argument similar to that given in Lemma 3.4 shows that $v_1 < z_1$ in $\mathbb{R} \times (0,T_4)$. (See [24] for details.) Hence, $v < z_1$ in $\mathbb{R}^+ \times (0,T_4)$.

Let $\sigma_2(t)$ be defined by $z_1(\sigma_2(t),t) = a$. That is $\sigma_2(t) = \sigma(t) + x_0$. Note that $s(t) < \sigma_2(t)$.

Set $I_t = \{x:v(x,t) = a\}$. An argument similar to that given in Lemma 3.5 shows that $I_t = \{s(t)\}$, and $s(t)$ is a continuous function as long as $s(t) > 0$. To show that $s(t) > 0$ in $(0,T_4)$ we consider the function $u(x,t)$ defined to be the solution of (3.36). For initial datum we assume that $u(x,0) = \psi(x)$, where $\psi(x)$ satisfies (1.2) with $x_0 > \theta_1$. Furthermore, we assume that

(a) $0 \leq \psi(x) \leq \varphi(x)$ in $\mathbb{R}^+$,

(b) $\psi(x) > a, \quad 0 < x < x_0 - 1$,

(c) $\psi(x_0 - 1) = a$.

We assume that $\theta > \theta_1 + 1$. Therefore, if $x_0 > \theta$, it follows that the curve $\sigma_1(t)$, defined by $u(\sigma_1(t),t) = a$, is a well defined, smooth function such that $\lim_{t\to\infty} \sigma_1(t) = \infty$, $\sigma_1(t) > x_0 - \beta$ in $\mathbb{R}^+$, $u(x,t) > d$ for $x < x_0 - \beta$, $t > T_3$, and $\sigma_1(T_4) \geq x_0 + \beta_1$. A proof similar to that given in Lemma 3.6 of (E) shows that $s(t) > \sigma_1(t)$ in $(0,T_4)$. Therefore,

(a) $s(t) > x_0 - \beta$ in $(0,T_4)$,

(3.38) (b) $s(T_4) > x_0 + \beta_1$,

(c) $v(x,t) > d$ for $x \in (0,x_0 - \beta)$, $t > T_3$.

Let $T_0 = \inf\{t:s(t) = x_0 + \beta_1\}$.

Recall that $s(t) < \sigma_2(t) = \sigma(t) + x_0$, and $\beta_1 = \sup_{0\leq t\leq T_3} \sigma(t)$. Therefore $s(t) < x_0 + \beta_1$ for $t \in (0,T_3)$. Since $s(T_0) = x_0 + \beta_1$ it follows that $T_3 < T_0$. The proof of Theorem 3.2 is now split into a number of Lemmas.

Lemma 3.7: $|v(x,T_0)| < |G_{s(T_0)}(x)|$ for $x > s(T_0)$.

Proof: Let $G = \{(x,t):x > s(t),\ t \in (0,T_0)\}$. We first show that $v > -a$ in G. Note that $v(s(t),t) = a$, and $v(x,0) > -a$ for $x > x_0$. It was shown earlier that $w < m$ in G. Therefore, $Lv = -w > -m > -a = L(-a)$ in G. From Theorem 2.1 it now follows that $v > -a$ in G. From the definition of $s(t)$ we conclude that $v < a$ in G, and, therefore, $|v| < a$ in G.

We now apply Proposition 2.5 with $v_0(x,t) = ae^{\frac{\sqrt{2}}{2}(s(T_0)-x)}$,

$v_1(x,t) = -ae^{\frac{\sqrt{2}}{2}(s(T_0)-x)}$, and $\alpha_0(x) \equiv \alpha_1(x) \equiv 0$ to conclude that

$|v(x,t)| < ae^{\frac{\sqrt{2}}{2}(s(T_0)-x)}$ for $x > s(T_0)$, $t \in (0,T_0)$. Therefore,

$|v(x,T_0)| < ae^{\frac{\sqrt{2}}{2}(s(T_0)-x)} = |G_{s(T_0)}(x)|$ for $x > s(T_0)$.

<u>Lemma 3.8</u>: If ε is sufficiently small, then $|w(x,T_0)| < H_{s(T_0)}(x)$ for $x > s(T_0)$.

<u>Proof</u>: Assume that

$$\varepsilon < \frac{m}{2T_4} .$$

Since $w(x,0) \equiv 0$, it follows from (3.6) that, for $x > s(T_0)$,

$$\begin{aligned}|w(x,T_0)| &< \varepsilon \int_0^{T_0} |v(x,\eta)|\,d\eta \\ &< \varepsilon T_4 a e^{\frac{\sqrt{2}}{2}(s(T_0)-x)} \\ &< \frac{m}{2} a e^{\frac{\sqrt{2}}{2}(s(T_0)-x)} \\ &= H_{s(T_0)}(x) .\end{aligned}$$

<u>Lemma 3.9</u>: If ε is sufficiently small then $|w(x,T_0)| < H_{s(T_0)}(x)$ for $x \in [0,s(T_0))$.

<u>Proof</u>: Assume that

$$\varepsilon < \frac{m}{2VT_4} .$$

Since $w(x,0) = 0$, it follows from (3.6) that, for $x \in [0,s(T_0))$,

$$|w(x,T_0)| < \epsilon \int_0^{T_0} |v(x,\eta)|d\eta < \epsilon VT_4 < \frac{m}{2} < H_{s(T_0)}(x) .$$

__Lemma 3.10:__ Let $P = \{(x,t): 0 < x < s(t), 0 < t < T_0\}$. Then $v(x,t) > a$ in P.

__Proof:__ Suppose that this is not true. From the results of [23] there must exist a point $(y,t_1) \in P$ such that $v(y,t_1) = a$ and $v(x,t) < a$ in $P_1 = \{(x,t) \in P: 0 < t < t_1\}$. We use a comparison argument to show that this is impossible.

Note that $Lv = 1 - w > 1 - m > a = La$ in P_1. Furthermore, $v(x,0) > a$ in $(0,x_0)$, $v(s(t),t) = a$ in $(0,t_1)$, and $v_x(0,t) = 0$ in $(0,t_1)$. From Theorem 2.1 it follows that $v(y,t_1) > a$ which is a contradiction.

The following result completes the proof of Theorem 3.2.

__Lemma 3.11:__ $G_{s(T_0)}(x) < v(x,T_0)$ in $[0,s(T_0))$.

__Proof:__ Since $G_{s(T_0)}(x) = a$ for $x \in (s(T_0) - \lambda_1, s(T_0))$, the previous lemma implies that $G_{s(T_0)}(x) < v(x,T_0)$ for $x \in (s(T_0) - \lambda_1, s(T_0))$. Recall, from (3.38c), that $v(x,T_0) > d$ for $x \in [0, x_0 - \beta]$. Since $G_{s(T_0)}(x) \leq d$ for all x the proof will be complete if we can show that $s(T_0) - \lambda_1 < x_0 - \beta$. However, since $\lambda_1 > C_3$ (see (3.37)) it follows that

$$s(T_0) - \lambda_1 = x_0 + \beta_1 - \lambda_1 < x_0 + \beta_1 - C_3 = x - \beta .$$

APPENDIX

Here we present a precise description of the comparison functions $G_{x_0}(x)$ and $H_{x_0}(x)$, and the various constants mentioned in the statement of Theorems 1.1, 3.1, and 3.2. These functions and constants are defined in their correct, logical order without any attempt at motivation. We shall see that these constants and functions depend only on the parameters a and γ.

Set $m = \frac{1}{4}\min(\frac{1}{2} - a, a)$ and $\lambda_3 = \log[2(\frac{1-(a+m)}{m})]$. Define $g_2(x)$ to be the solution of the differential equation

$$g_2'' - g_2 = m - 1 \qquad \text{in } (-\lambda_3, 0),$$
$$g_2(-\lambda_3) = a, \qquad g_2'(0) = 0 .$$

Note that $g_2(x) = [\frac{a+m-1}{e^{\lambda_3} + e^{-\lambda_3}}](e^x + e^{-x}) + 1 - m$. Let $d = g_2(0)$.

Recall from the introduction that there exist constants V and W such that $|v(x,t)| < V$ and $|w(x,t)| < W$ in $\mathbb{R} \times \mathbb{R}$. Let $\xi_1(x)$ be a smooth function which satisfies (3.35). Let $z_1(x,t)$ be the solution of the differential equation:

$$z_{1t} = z_{1xx} + f(z_1) + m \qquad \text{in } \mathbb{R} \times \mathbb{R}^+,$$
$$z_1(x,0) = \xi_1(x) \qquad \text{in } \mathbb{R} .$$

Define $\sigma_1(t)$ by $z_1(\sigma_1(t),t) = a$, $\sigma_1(0) = x_0$. A slight modification of the proof of Theorem 2.4 shows that $\sigma_1(t)$ is a smooth function such that $\lim_{t\to\infty} \sigma_1(t) = \infty$. Note that we cannot apply Theorem 2.4 directly because $\xi_1(x) \neq \xi_1(-x)$ in $\mathbb{R}^+$. The proof of Theorem 2.4 is easier, however, with the assumption (3.35c). Let $T_1 = \inf\{t : \sigma_1(t) = 1\}$.

From Theorem 2.4 we conclude that there exists a constant C_0 with the following property. Suppose that $y > C_0$ and $z(x,t)$ satisfies the equations:

$$z_t = z_{xx} + f(z) + m \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^+$$
$$z_x(0,t) = 0 \qquad \text{in } \mathbb{R}^+$$
$$z(x,0) = \xi(x) \qquad \text{in } \mathbb{R}^+ .$$

Here $\xi(x)$ is any smooth function which satisfies

(a) $\xi(x) \in (V, 2V)$ for $0 < x < y$,

(b) $\xi(x) \in C^2(\mathbb{R}^+)$,

(c) $\xi'(x) < 0$ in $\mathbb{R}^+$,

(d) $\xi(x) = ae^{\frac{\sqrt{2}}{2}(y + \frac{1}{2} - x)}$ for $x > y + \frac{1}{2}$.

Then the function $\sigma(t)$, given by $z(\sigma(t),t) = a$ is a well defined, smooth function which satisfies $\lim_{t\to\infty} \sigma(t) = \infty$.

Let $\bar{\lambda}_2 = \frac{2}{1-(d+m)} [\frac{1+T_1}{T_1}](d-a)$, and $\lambda_2 = \bar{\lambda}_2 + \lambda_3$. Let

$$g_1(x) = \begin{cases} d & \text{for } 0 \leq x \leq \lambda_3 \\ d - [\frac{d-a}{-\bar{\lambda}_2^2}](x-\lambda_3)^2 & \text{for } \lambda_3 < x \leq \lambda_2 . \end{cases}$$

Define, for $y > 0$ and $x \in [0,y]$, $g_3(x,y)$ to be the solution of the differential equation

$$g_3'' - g_3 = -\frac{m}{2}(x-y) + m \qquad \left(g_3' = \frac{\partial g_3}{\partial x}\right).$$

$$g_3'(0,y) = 0, \quad g_3(y,y) = a .$$

Then, $g_3(x,y) = [\frac{a + m - \frac{m}{2}e^{-y}}{e^y + e^{-y}}][e^x + e^{-x}] + \frac{m}{2}e^{-x} + \frac{m}{2}(x-y) - m$.

Let $g(x)$ be the solution of the differential equation:

$$g'' - g = m \text{ in } \mathbb{R}^+ ,$$

$$g(0) = a, \quad g'(0) = -\frac{1}{2} .$$

Then, $g(x) = [\frac{a + m - \frac{1}{2}}{2}][e^x - e^{-x}] + [a+m]e^{-x} - m$. Note that $g'(x) < 0$ in $\mathbb{R}^+$ and $\lim_{x\to\infty} g(x) = -\infty$. Define C_2 by $g(C_2) = -a$.

Since $a + 2m < \frac{1}{2}$, we conclude from Theorem 2.4 that there exist constants C_1 and T_2 with the following properties. Suppose tht $y > C_1$ and let $\psi(x)$ be any smooth function which satisfies:

(a) $\psi(x) \in C^2(\mathbb{R}^+)$,

(b) $\psi(x) \in (-a, 1 - 2m)$ in $\mathbb{R}^+$,

(c) $\psi(x) > a$ for $x \in (0,y)$,

(d) $\psi'(x) < 0$ in $\mathbb{R}^+$,

(e) $\psi(y) = a$.

Let $u(x,t)$ be the solution of the differential equation

$$\begin{aligned} u_t &= u_{xx} + f(u) - 2m && \text{in } \mathbb{R}^+ \times \mathbb{R}^+ , \\ u_x(0,t) &= 0 && \text{in } \mathbb{R}^+ , \\ u(x,0) &= \psi(x) && \text{in } \mathbb{R}^+ . \end{aligned}$$

Then the function $\sigma_2(t)$, given by $u(\sigma_2(t),t) = a$, $\sigma_2(0) = y$, is a well defined, smooth function which satisfies:

(a) $\lim\limits_{t\to\infty} \sigma_2(t) = \infty$,

(b) $\sigma_2(t) > \lambda_2 + 1$ in $\mathbb{R}^+$,

(c) $\sigma_2(T_2) > y + 2$.

From Theorem 2.4 it follows that there exist constants θ_1, β, and T_3 with the following properties. Let $u_1(x,t)$ be the solution of the equations

$$\begin{aligned} u_{1t} &= u_{1xx} + f(u_1) - m && \text{in } \mathbb{R}^+ \times \mathbb{R}^+ , \\ u_{1x}(0,t) &= 0 && \text{in } \mathbb{R}^+ \end{aligned}$$

with initial datum $u_1(x,0) = \eta(x)$ which satisfies (1.2) with $x_0 > \theta_1$. Then the curve $\sigma_3(t)$, given by $u_1(\sigma_3(t),t) = a$, is a well defined smooth function which satisfies:

(a) $\lim\limits_{t\to\infty} \sigma_3(t) = \infty$,

(b) $\sigma_3(t) > x_0 - \beta$ in $\mathbb{R}^+$,

(c) $u_1(x,t) > d$ for $x < x_0 - \beta$, $t > T_3$.

Let $\beta_1 = \sup\limits_{0<t<T_3} \sigma_1(t)$ and $C_3 = \beta + \beta_1$. From Theorem 2.6 there exists a constant T_4 such that $\sigma_3(t) > x_0 + \beta_1$ for $t > T_4$.

Let $\lambda_1 = \max\{C_0, C_1, C_2 C_3\}$.

Choose $\lambda_4 > 0$ so that $e^{\lambda_4} - e^{-\lambda_4} = 1$, and let $\theta = \theta_1 + \sum_{k=1}^{4} \lambda_k$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

$$\delta_1 = \frac{m}{2VT_2(\lambda_1 + \lambda_2 + \lambda_3)} ,$$

$$\delta_2 = \frac{m}{2VT_2}\left(1 - e^{-\frac{\sqrt{2}}{2}}\right) ,$$

$$\delta_3 = \frac{m}{2VT_4} ,$$

$$\delta_4 = \frac{1}{VT_2} \min_{0<x<1} \left(\frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)}(1 - x) + \frac{m}{2} - \frac{m}{2} e^{-\frac{\sqrt{2}}{2} x}\right) .$$

Finally, let

$$G_{x_0}(x) = \begin{cases} -ae^{\frac{\sqrt{2}}{2}(x_0 - x)} & x_0 < x \\ a & x_1 \equiv x_0 - \lambda_1 < x < x_0 \\ g_1(x - x_2) & x_2 \equiv x_1 - \lambda_2 < x < x_1 \\ g_2(x - x_2) & x_3 \equiv x_2 - \lambda_3 < x < x_2 \\ g_3(x, x_3) & 0 < x < x_3 \end{cases}$$

$$H_{x_0}(x) = \begin{cases} \frac{m}{2} e^{\frac{\sqrt{2}}{2}(x_0 - x)} & x_0 < x \\ \frac{m}{2(\lambda_1 + \lambda_2 + \lambda_3)}(x_0 - x) + \frac{m}{2} & 0 < x < x_0 . \end{cases}$$

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We consider the pure initial value problem for the system of equations

$$v_t = v_{xx} + f(v) - w$$

$$w_t = \varepsilon(v - \gamma w), \quad \varepsilon, \gamma \geq 0 ,$$

the initial data being $(v(x,0), w(x,0)) = (\varphi(x), 0)$. Here $f(v) = -v + H(v - a)$

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20. ABSTRACT - cont'd.

where H is the Heaviside step function and $a \in (0,\frac{1}{2})$. This system is of the FitzHugh-Nagumo type, and has several applications including nerve conduction and distributed chemical/biochemical systems. It is demonstrated that this system exhibits a threshold phenomenon. This is done by considering the curve $s(t)$ defined by $s(t) = \sup\{x : v(x,t) = a\}$. The initial datum, $\varphi(x)$, is said to be superthreshold if $\lim_{t\to\infty} s(t) = \infty$. It is proven that the initial datum is superthreshold if $\varphi(x) > a$ on a sufficiently long interval, $\varphi(x)$ is sufficiently smooth, and $\varphi(x)$ decays sufficiently fast to zero as $|x| \to \infty$.

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